Measures in ternary homogeneous structures

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① Measures on countable graphs and hypergraphs

Measures are important in model theory since

- they give a good notion of size on infinite structures;
- they often capture how a structure can be obtained by random constructions (e.g. the **random graph**);
- they naturally arise in infinite structures approximating the behaviour of a class of finite structures (e.g. **pseudofinite fields**).

We focus on probability measures arising in highly symmetrical countable **graphs** and **hypergraphs**.

The 3-hypergraphs that we study exhibit oddly-behaved measures which serve as counterexamples to many conjectures in the field.

② Definable sets in graphs and hypergraphs

We focus on **definable sets** in graphs and hypergraphs. For example, for a graph G and $a, b \in G$,

$$E(x,a) \wedge E(x,b)$$

defines the set of vertices $x \in G$ joined to both a and b. A **3-hypergraph** has a 3-hyperedge relation $R(x,y,z)$.

We work with **homogeneous** structures: any isomorphism between finite substructures extends to an automorphism of the whole structure.

③ Invariant Keisler measures

A **Keisler measure** on a structure $\mathcal{M}$ is a finitely additive **probability** measure on its definable subsets. We are interested in measures **invariant under automorphisms**:

$$\mu(X) = \mu(\sigma(X)) \text{ for } \sigma \in \mathrm{Aut}(M).$$

For $d \in M, A \subset M, \mu(d/A)$ is the measure of the intersection of all sets containing d defined with parameters from A.

④ Example: measures in homogeneous graphs

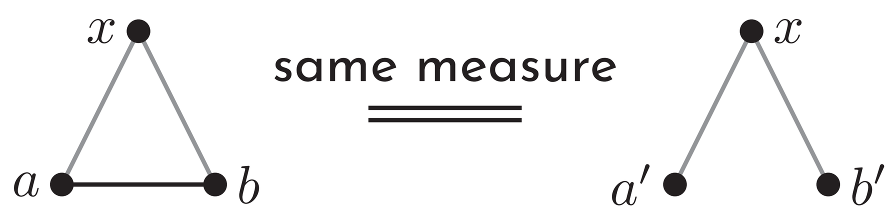
Countable homogeneous graphs have been classified in [5]. Two important examples:

- the **random graph**. The unique countable homogeneous graph containing every finite graph as a subgraph;
- the **generic triangle-free graph**. The unique countable homogeneous graph containing every finite triangle-free graph.

In a countable homogeneous graph, the measure $\mu(d/A)$ **does not depend on the structure of A**

For example, for $a, b \in G$ forming an edge and $a', b' \in G$ not forming an edge,

$$\mu(E(x,a) \wedge E(x,b)) = \mu(E(x,a') \wedge E(x,b')).$$



This is due to a very general and powerful principle that holds in this context [4]:

Stationarity: Given a and b independent,

$$\mu(\phi(x,a) \wedge \psi(x,b))$$

only depends on the structure of a and b respectively, and not on the structure of ab.

Stationarity and some ergodic theory allow for a full classification of invariant Keisler measures on homogeneous countable graphs [1].

⑤ Higher stationarity?

Given a, b, c satisfying adequate independence assumptions, can we prove that

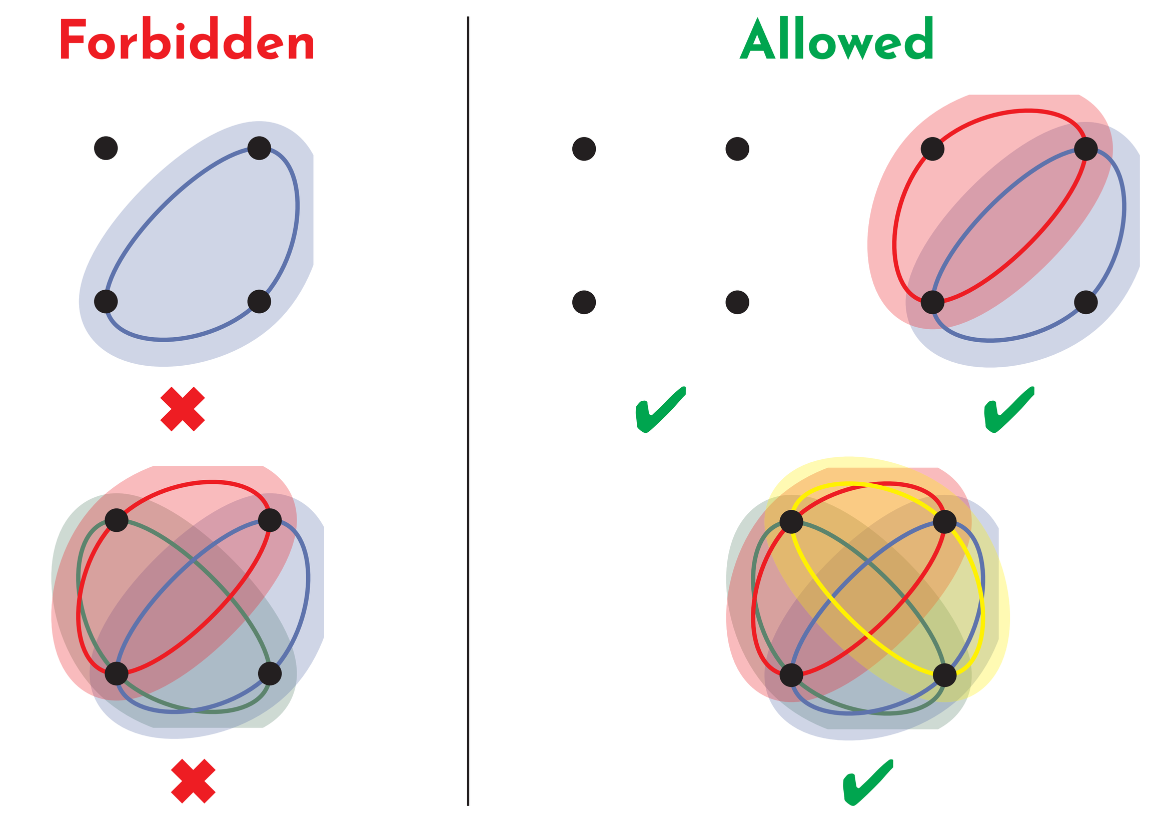
$$\mu(\phi(x,ab) \wedge \psi(x,ac) \wedge \chi(x,bc))$$

only depends on the structure of the pairs ab, ac, bc and not on the structure of the whole triplet?

A version of this **higher stationarity** holds in many structures (e.g. homogeneous graphs). Moreover it was suggested by hypergraph regularity results in pseudofinite fields [2].

⑥ The universal homogeneous two-graph has a unique measure

A **two-graph** is a 3-hypergraph such that every four vertices have an even number of hyperedges.



The **universal homogeneous two-graph** $\mathcal{G}$ is the unique countable 3-hypergraph containing every finite two-graph as a substructure.

A unique measure

There is a unique invariant Keisler measure for $\mathcal{G}$. Given $d, a_1, \dots, a_n$ distinct, we have that

$$\mu(d/a_1, \dots, a_n) = \left(\frac{1}{2}\right)^{n-1}.$$

The universal homogeneous two-graph gives **a counterexample to higher stationarity:**

Take $a, b, c \in \mathcal{G}$ not forming a hyperedge. Then,

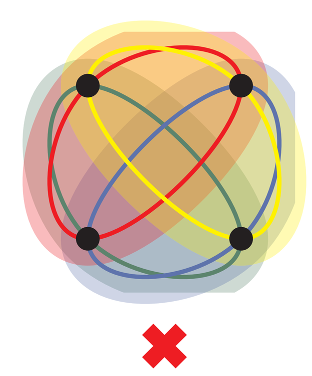
$$\mu(R(x,ab) \wedge R(x,ac) \wedge R(x,bc)) = 0,$$

because if there was any such x, there would be four vertices with three hyperedges. Meanwhile, for $a', b', c' \in \mathcal{G}$ forming a hyperedge,

$$\mu(R(x,a'b') \wedge R(x,a'c') \wedge R(x,b'c')) = \frac{1}{4}.$$

⑦ The generic tetrahedron-free 3-hypergraph is not MS-measurable

A **tetrahedron** is a 3-hypergraph of four vertices such that each three of them form a hyperedge.



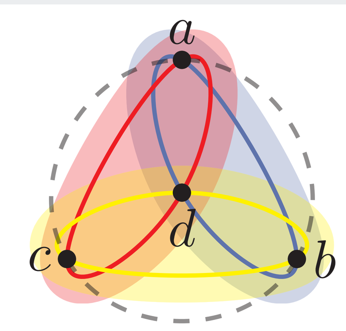
The **generic tetrahedron-free 3-hypergraph** $\mathcal{H}$ is the unique countable homogeneous 3-hypergraph containing every finite tetrahedron-free 3-hypergraph as a substructure.

An MS-measurable structure [6] has a well-behaved system of a dimension and measures. It was believed that various properties satisfied by $\mathcal{H}$ jointly implied MS-measurability [3]. However,

No "well-behaved" system of measures

The generic tetrahedron-free 3-hypergraph is not MS-measurable.

The trick behind the proof is **forcing higher stationarity**: fix d and consider a, b, c such that each pair forms a hyperedge with it. abc cannot form a hyperedge since we avoid tetrahedrons!



Hence, for any formulas, and μ $\mathrm{Aut}(H/d)$-invariant:

$$\mu(\phi(x,ab,d) \wedge \psi(x,ac,d) \wedge \chi(x,bc,d))$$

is entirely determined by the structure of each triplet containing d and two vertices from abc.

⑧ References

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[3] R. Elwes and D. Macpherson. "A survey of asymptotic classes and measurable structures". In: *Model Theory with Applications to Algebra and Analysis*. Vol. 2. London Mathematical Society Lecture Note Series. CUP, 2008, pp. 125–160.

[4] E. Hrushovski. "Stable Group Theory and Approximate subgroups". In: *J Am Math Soc* 25.1 (2012), pp. 189–243.

[5] A. H. Lachlan and R. E. Woodrow. "Countable Ultrahomogeneous Undirected Graphs". In: *Trans Am Math Soc* 262.1 (1980), pp. 51–94.

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