Invariant Keisler Measures in ω-categorical Hrushovski constructions

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① Two notions of smallness

Working in a structure $\mathcal{M}$, there are two natural ways in which we may say that the set defined by the formula $\phi(x,a)$ is "small":

- $F(\emptyset)$ $\phi(x,a)$ **forks** over $\emptyset$. Call $F(\emptyset)$ the set of such formulas;
- $\mathcal{O}(\emptyset)$ $\phi(x,a)$ is **universally measure zero**. i.e. it has measure zero for any invariant Keisler measure. Call $\mathcal{O}(\emptyset)$ the set of such formulas;

For stable theories, $F(\emptyset)=\mathcal{O}(\emptyset)$. This should also be the case for NIP theories.

Until the recent counterexamples from [1], it was an open question whether this equality always holds in **simple** theories. It is natural to ask whether the equality holds for **ω-categorical simple** structures.

I proved that for various classes of supersimple ω-categorical Hrushovski constructions $F(\emptyset)\subsetneq\mathcal{O}(\emptyset)$. These are the first known **ω-categorical** examples of this phenomenon.

② Invariant Keisler Measures

A **Keisler measure** on M is a finitely additive **probability** measure on its definable subsets.
We are interested in measures **invariant under automorphisms**:

$$\mu(X)=\mu(\sigma(X)) \text{ for } \sigma\in\mathrm{Aut}(M).$$

There is a correspondence between Keisler measures and regular Borel probability measures on the space $S_x(M)$.

The measure μ is **ergodic** if for any Borel A,

$$\mu(A\triangle\sigma(A))=0 \quad \forall\sigma\in\mathrm{Aut}(M)\Rightarrow\mu(A)=0 \text{ or } 1.$$

Ergodic measures are better behaved, and yield an **ergodic decomposition** of any invariant Keisler measure:

$$\mu(A)=\int_{\mathrm{Erg}(M)}\nu(A)\mathrm{d}m(\nu).$$

③ Weak Algebraic Independence and Probabilistic Independence

We say that $A,B\subseteq\mathcal{M}^{eq}$ are **weakly algebraically independent** if $\mathrm{acl}^{eq}(A)\cap\mathrm{acl}^{eq}(B)=\mathrm{acl}^{eq}(\emptyset)$. We write $A\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}B$.

For ω-categorical structures, weak algebraic independence induces a form of **probabilistic independence** when looking at ergodic measures:

Probabilistic independence theorem [4]

Let $\mathcal{M}^{eq}$ be ω-categorical with $\mathrm{acl}^{eq}(\emptyset)=\mathrm{dcl}^{eq}(\emptyset)$. Let μ be an ergodic measure and a,b be weakly algebraically independent. Then, for any formulas $\phi(x,y),\psi(x,z)$,

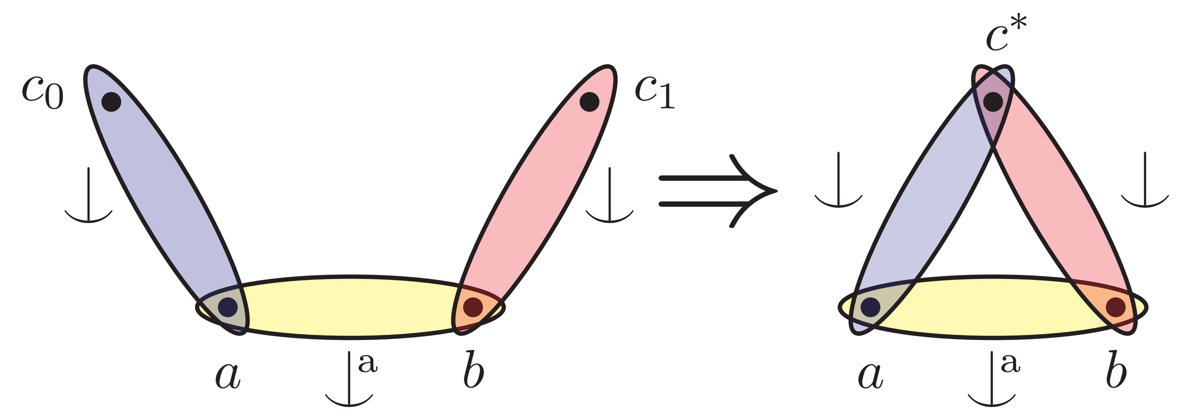
$$\mu(\phi(x,a)\wedge\psi(x,b))=\mu(\phi(x,a))\mu(\psi(x,b)).$$

Recently, [2] have generalised these results outside of the ω-categorical context.

④ Strong Independence Theorem

For simple structures, the Probablistic Independence Theorem yields a **stronger version of the independence theorem** over $\emptyset$ when **forking is the same as being universally measure zero**:

Say a and b are **weakly algebraically independent**, $c_0\equiv c_1$ and $c_0\mathop{\smile\!\!\!\!\!\!|}a$, $c_1\mathop{\smile\!\!\!\!\!\!|}b$. Then, there is c^* such that $c^*\equiv_a c_0$, $c^*\equiv_b c_1$, $c^*\mathop{\smile\!\!\!\!\!\!|}ab$.



In general, simple ω-categorical structures with $\mathrm{acl}^{eq}(\emptyset)=\mathrm{dcl}^{eq}(\emptyset)$ satisfy this for $a\mathop{\smile\!\!\!\!\!\!|}b$. But in our result we have weak algebraic independence instead of non-forking independence.

⑤ Measures in ω-categorical Hrushovski constructions

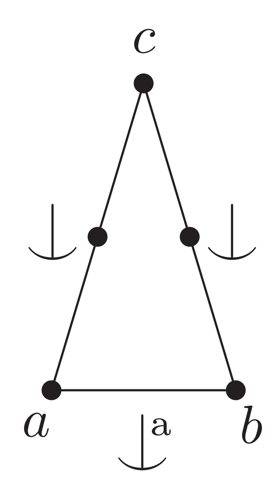
Q: Are there simple ω-categorical structures with $F(\emptyset)\subsetneq\mathcal{O}(\emptyset)$?
Idea for a counterexample: A simple ω-categorical structure which does not satisfy the strong independence theorem.
Candidate: simple ω-categorical Hrushovski constructions.
Why? They are the only known example of supersimple ω-categorical **not one-based** structures (i.e. weak algebraic independence $\neq$ non-forking independence). So we may be able to construct simple ones not satisfying the strong independence theorem (and indeed we are!).

In particular, we build an ω-categorical supersimple Hrushovski construction $\mathcal{M}$ of SU-rank 2, which is a **graph** such that:

- $\mathrm{acl}^{eq}(\emptyset)=\mathrm{dcl}^{eq}(\emptyset)$.
- $\mathrm{Aut}(M)$ acts transitively in the vertices of M.
- There are no k-cycles for $k<6$.
- If a,b form an edge, $a\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}b$ (but not $a\mathop{\smile\!\!\!\!\!\!|}b$).
- If a and c are at distance two from each other, then $a\mathop{\smile\!\!\!\!\!\!|}c$.
- The formula $\phi(x,a)$ saying "x has distance two from a" doesn't fork over the empty-set.

We can also build $\mathcal{M}$ witnessing arbitrarily strong independent n-amalgamation properties.

In the way we built our graph, we can see that for $\mathcal{M}$ to satisfy the strong independence theorem, it **should contain pentagons!**



But $\mathcal{M}$ has no pentagons and so:

$F(\emptyset)\subsetneq\mathcal{O}(\emptyset)$ in ω-categorical simple structures

There are ω-categorical simple structures with $F(\emptyset)\subsetneq\mathcal{O}(\emptyset)$. In particular, various ω-categorical supersimple Hrushovski constructions witness this.

⑥ Non-MS-measurability

An **MS-measurable structure** has a **dimension-measure** function which is **definable** and **finite** and where the dimension and the associated measures satisfy **Fubini's theorem** [5].

Elwes and Macpherson [3] asked whether all ω-categorical supersimple structures of finite SU-rank are MS-measurable.

Supersimple ω-categorical finite rank and NOT MS-measurable

The same example shows that various ω-categorical Hrushovski constructions are not MS-measurable. In fact, for ω-categorical MS-measurable structures, $F(\emptyset)=\mathcal{O}(\emptyset)$.

⑦ Ongoing work

Recently, I proved that satisfying the strong independence theorem does not imply $F(\emptyset)=\mathcal{O}(\emptyset)$.

SIT $\not\Rightarrow F(\emptyset)=\mathcal{O}(\emptyset)$

There are supersimple ω-categorical Hrushovski constructions satisfying the strong independence theorem but still with $F(\emptyset)\subsetneq\mathcal{O}(\emptyset)$.

Open questions:

(1) Is there any supersimple ω-categorical not one-based Hrushovski construction for which $F(\emptyset)=\mathcal{O}(\emptyset)$ (perhaps even MS-measurable)?

(2) Is every ω-categorical MS-measurable structure one-based?

⑧ References

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[4] C. Jahel and T. Tsankov. "Invariant measures on products and on the space of linear orders". In: *Journal de l'École polytechnique — Mathématiques* 9 (2022), pp. 155-176.

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