

All mixed identities are singular in groups with no algebraicity

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joint work with Michael Pinsker

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Mixed identities

Outline

- ① Background
 - Equations in groups
 - Mixed identities
- ② On algebraic closure
- ③ The result
- ④ Strategy
- ⑤ A topological extra
- ⑥ Bibliografy

Words with constants

G := a group.

Definition (Word with constants)

A **word with constants** for G in r many variables is

$$w(x_1, \dots, x_r) := \gamma_n x_{\iota_n}^{\epsilon_n} \gamma_{n-1} x_{\iota_{n-1}}^{\epsilon_{n-1}} \dots \gamma_1 x_{\iota_1}^{\epsilon_1} \gamma_0$$

for $\gamma_0, \dots, \gamma_n \in G$, $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$, and $\iota_1, \dots, \iota_n \in \{1, \dots, r\}$.

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 I.e., $w \in G * F_r$.

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 I.e., a **term** in $\mathcal{L} := \{\cdot, {}^{-1}, (\gamma)_{\gamma \in G}\}$.

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w is **singular** if “forgetting constants” and reducing we get 1:

$$y^{-1} x \gamma_2 x^{-1} \gamma_1 y \xrightarrow[\text{constants}]{\text{forget}} y^{-1} x x^{-1} y \mapsto y^{-1} y \mapsto 1.$$

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Otherwise, w is **regular** (a.k.a. non-singular).

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- when can we **solve** w ?

$$\exists g_1, \dots, g_r \in G \text{ s.t. } w(g_1, \dots, g_r) = 1 ;$$

- when is w a **mixed identity**?

$$\forall g_1, \dots, g_r \in G \ w(g_1, \dots, g_r) = 1 .$$

👁 These notions are in tension!

$$\forall \bar{g} \in G \ w(\bar{g}) = 1 \Rightarrow \text{for } \gamma \in G \setminus \{1\}, \nexists \bar{g} \in G \ \gamma w(\bar{g}) = 1.$$

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Solving equations

Conjecture (Klyachko and Thom 2017)

G a group, $w(x_1, \dots, x_r)$ **regular**.

Then, there is $H \supseteq G$ s.t. $\exists h_1, \dots, h_r \in H$ $w(h_1, \dots, h_r) = 1$.

- $r = 1$ this is the Kervaire-Laudenbach Conjecture:
 - Gerstenhaber and Rothaus 1962: true for all finite groups;
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 - OpenAI 2026b  (presumably?): true for all groups.*

*OpenAI 2026a implies a weaker version of the Klyachko-Thom conjecture.

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$G := \text{Aut}(\mathbb{Q}; <)$. w *with no constants and non-trivial*.

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Theorem (Mycielski 1987; Lyndon 1990)

$G := \text{Sym}(\mathbb{N})$. w *with no constants, non-trivial, and $w \neq x^n$ ($n \geq 2$)*.

Then, $w(G^r) = G$.

An interlude on oligomorphicity

$\text{Aut}(\mathbb{Q}; <)$ and $\text{Sym}(\mathbb{N})$ are both oligomorphic:

Definition (Oligomorphic)

$|\Omega| = \aleph_0$. $G \curvearrowright \Omega$ is **oligomorphic** if for all $n \in \mathbb{N}$, $G \curvearrowright \Omega^n$ has finitely many orbits in the action $g \cdot (a_1, \dots, a_n) \mapsto (ga_1, \dots, ga_n)$.

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Ryll-Nardzewski: $G = \text{Aut}(M)$ oligomorphic $\Leftrightarrow M$ is ω -categorical.*

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 Algebraic examples: $\text{GL}(\aleph_0, \mathbb{F}_q)$, $\text{Sp}(\aleph_0, \mathbb{F}_q)$, $\text{U}(\aleph_0, \mathbb{F}_q)$, $\text{O}(\aleph_0, \mathbb{F}_q)$.

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Fraïssé:

$$\left\{ \begin{array}{l} \text{classes of finite substructures (} \mathbf{ages} \text{) of} \\ \text{homogeneous structures} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Fraïssé} \\ \text{classes} \end{array} \right\}.$$

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Fraïssé:* hereditary class $\mathcal{C}$ with the **amalgamation property**:

(AP) for $A, B_0, B_1 \in \mathcal{C}$, and embeddings $f_i : A \rightarrow B_i$ ($i \in \{0, 1\}$), there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \rightarrow C$ ($i \in \{0, 1\}$) s.t. $g_1 \circ f_1 = g_2 \circ f_2$.

*Assuming language is countable and purely relational.

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Often, we specify homogeneous structures by their age:
 random graph, generic triangle-free graph, random poset. . .

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Mixed identities

Definition (MIF and lawless)

$w(x_1, \dots, x_r)$ is a **mixed identity** if $w(G^r) = 1$.

A mixed identity without constants is a **law**.

G is **mixed-identity-free (MIF)**: no (non-trivial) mixed identities.

G is **lawless**: no (non-trivial) laws.

👁 Mixed identities describe the structure of a group/elements:

$$\forall x \ x^2 = 1 \quad \text{or} \quad \forall x \ x\gamma x^{-1}\gamma^{-1} = 1$$

💡 we would expect “rich group” $\Rightarrow$ “few” mixed identities.

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Theorem (Macpherson 1986)

$G \curvearrowright \Omega$ *oligomorphic* $\Rightarrow G$ is lawless.

Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space

(Etedadialiabadi, Gao, Le Maître, and Melleray 2021);

- transitive with free amalgamation

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Sometimes we have mixed identities!

- $\text{Sym}(\mathbb{N})$: $[x, t]^6$ where t is a transposition
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(Ghadernezhad and de la Nuez González 2019; Bodirsky, Schneider, and Thom 2025);
- random poset, random permutation
(Bodirsky, Schneider, and Thom 2025).

Sometimes we have mixed identities!

- $\text{Sym}(\mathbb{N})$: $[x, t]^6$ where t is a transposition
(Hull and Osin 2016);
- $\text{GL}(\aleph_0, \mathbb{F}_q)$
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Mixed identities in the oligomorphic/homogeneous context

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In the first two cases, we know all mixed identities are singular!

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Conjecture (Bodirsky, Schneider, and Thom 2025)

$G \curvearrowright \Omega$ oligomorphic $\Rightarrow$ all mixed identities of G are singular.

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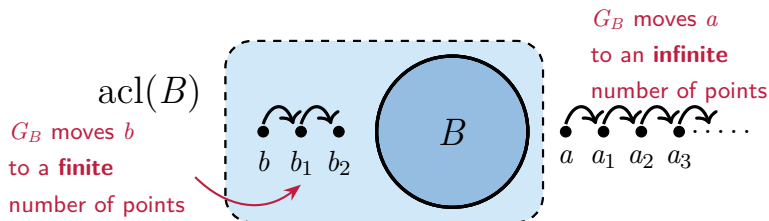
Which general conditions imply that all mixed identities are singular?

Algebraic closure

Definition (algebraic closure)

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$$\text{acl}(B) := \{a \in \Omega \mid a \text{ has finite } G_B\text{-orbit}\}.$$



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Theorem (Abért 2005)

$G \curvearrowright \Omega$ with no algebraicity. *Then, G is lawless.*

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Theorem (Bardyla, Elliott, Mitchell, and Péresse 2025)

$G \curvearrowright \Omega$ with no algebraicity. Then, for distinct **semigroup** words w, v **in one variable**, there is $g \in G$, s.t. $w(g) \neq v(g)$.

Our result

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 - recovers most previous results;
 - confirms the Bodirsky, Schneider, and Thom 2025 conjecture for several cases beyond previous techniques:
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💡 More precisely: the set of non-solutions to a regular mixed word

$$\{(g_1, \dots, g_r) \in G^r \mid w(g_1, \dots, g_r) \neq 1\}$$

is dense in G^r wrt the topology of pointwise convergence.[†]

👁 We do not need the action to be faithful.

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Proof strategy[‡]

For $g \in G$, $a_0 \in \Omega$, the images of a_0 with respect to subwords of w create a sequence $(a_0, a_1, a'_1, \dots, a_n, a'_n, a_{n+1})$ with

$$g \cdot (a_1, \dots, a_n) = (a'_1, \dots, a'_n) .$$

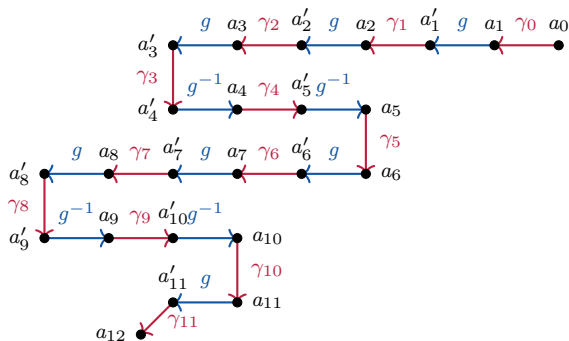
[‡] w with one variable.

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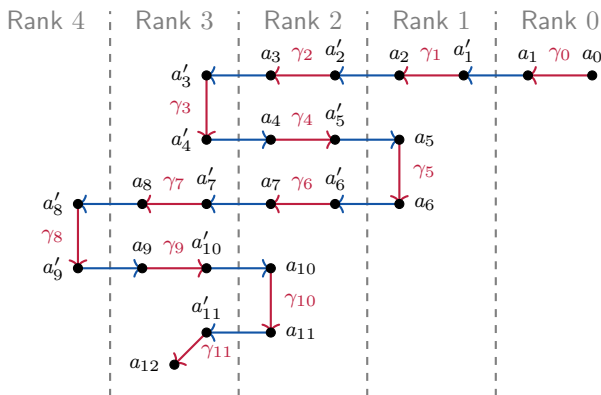


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Rank elements according to their position in $\mathbb{Z}$.

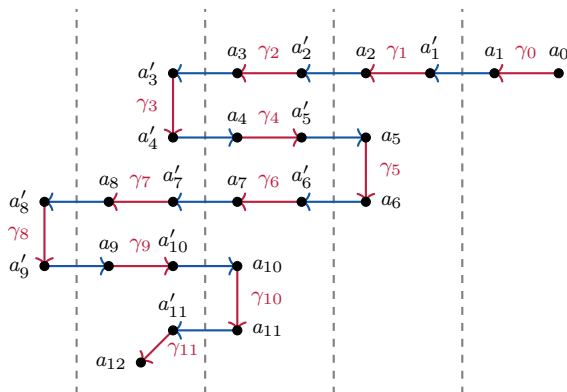


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We want g so that elements of different ranks are independent!

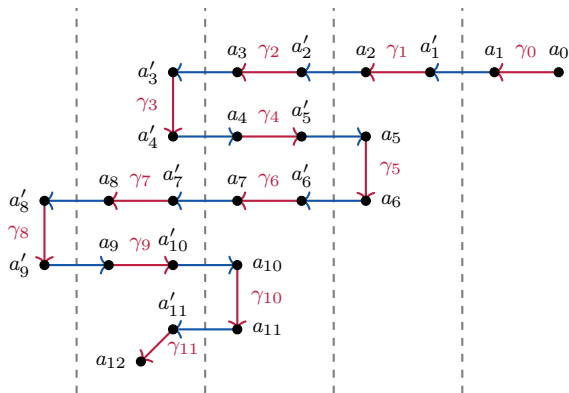


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We cannot guarantee elements of same rank to be distinct!

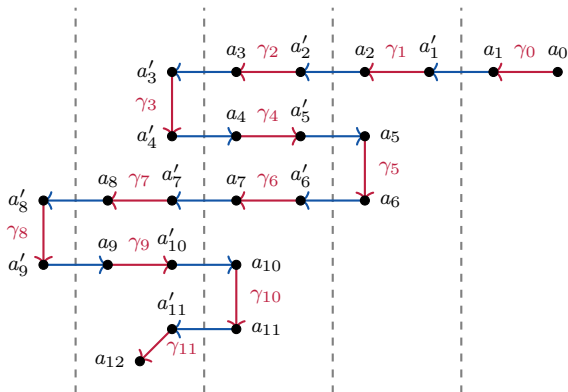


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Use **Neumann's Lemma** to guarantee independence (looking ahead!)

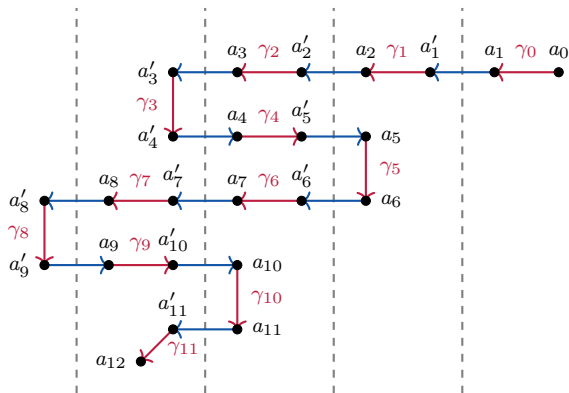


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Induction proof: challenge is balancing inductive hypothesis!



Geometric properties underlying the proof

With algebraicity, we need further geometric properties and a strengthening of Neumann's Lemma.

Definition (pregeometry)

(Ω, cl) for $\text{cl} : \mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$ is a **pregeometry** if it satisfies:

(REFLEXIVITY) $A \subseteq \text{cl}(A)$;

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(FINITE CHARACTER) $\text{cl}(A) = \bigcup \{\text{cl}(C) \mid C \subseteq A \text{ finite}\}$

(EXCHANGE) if $a \in \text{cl}(A \cup \{b\}) \setminus \text{cl}(A)$,
then $b \in \text{cl}(A \cup \{a\})$.

It is **G-invariant** if for $g \in G$, $\text{cl}(gA) = g \text{cl}(A)$.

(Ω, cl) is **modular** if for all closed $A, B \subseteq \Omega$,

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B).$$

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Higher Neumann pregeometries

Definition

Let $G \curvearrowright \Omega$. G -invariant (Ω, cl) is **higher Neumann** if for each $B, C \subseteq \Omega$ finite, $a \in \Omega$ such that $\text{cl}(a) \cap \text{cl}(B) = \text{cl}(\emptyset)$, and $\Sigma \subseteq G$ finite, there is $g \in G_B$ such that

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Let $G \curvearrowright \Omega$. Let (Ω, cl) be a G -invariant infinite-dimensional higher Neumann modular pregeometry.

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Applies to $\text{GL}(\kappa, K)$ and $\text{PGL}(\kappa, K)$ for $\kappa \geq \aleph_0$, recovering results of Bradford, Schneider, and Thom 2023.

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Then all mixed identities of G are singular.

Bradford, Ersoy, Schneider, and Thom 2026:

also yields the result for $\text{Sp}(\aleph_0, \mathbb{F}_q)$, $\text{SU}(\aleph_0, \mathbb{F}_q)$, $\text{O}(\aleph_0, \mathbb{F}_q)$.

Higher Neumann pregeometries

Definition

Let $G \curvearrowright \Omega$. G -invariant (Ω, cl) is **higher Neumann** if for each $B, C \subseteq \Omega$ finite, $a \in \Omega$ such that $\text{cl}(a) \cap \text{cl}(B) = \text{cl}(\emptyset)$, and $\Sigma \subseteq G$ finite, there is $g \in G_B$ such that

$$\text{cl} \left(\bigcup_{\gamma \in \Sigma} \gamma g(a) \right) \cap \text{cl}(C) = \text{cl}(\emptyset) .$$

Theorem (MP 2026)

Let $G \curvearrowright \Omega$. Let (Ω, cl) be a G -invariant infinite-dimensional higher Neumann modular pregeometry.

Then all mixed identities of G are singular.

► More on the proof

Microsupported actions

Several of the examples coming from topology are microsupported:

Definition

$\Omega = (\Omega; \tau)$ Hausdorff, $G \leq \text{Homeo}(\Omega)$. G is **microsupported** if for every non-empty open set $U \subseteq \Omega$, $G_{\Omega \setminus U} \neq \{1\}$.

Theorem (Nekrashevych 2022)

If G is microsupported, then G is lawless.

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If G is microsupported, then all of its mixed identities are singular.

Different proof idea: lifting lawlessness through microsupport.

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Thank you for your time!

Recap:

- All mixed identities being singular is a strong form of lawlessness;
- We prove this for:
 - groups with no algebraicity;
 - groups with microsupported actions.
- Key new idea: use global geometric properties of actions instead of properties of constants (controlling products of constants).

Paper:



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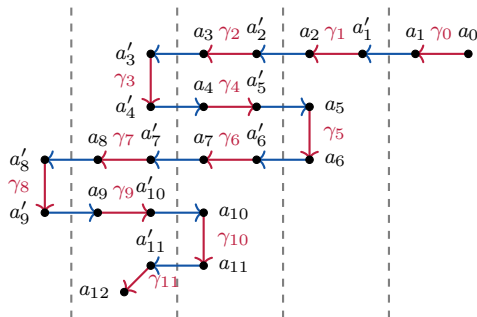
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Why these geometric conditions?

Induction splits into two cases:

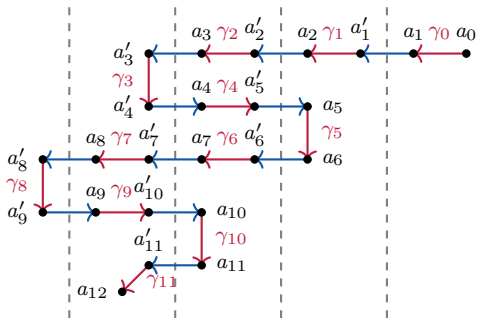
- a_{n+1} is in the closure of previous elements;
- a_{n+1} is not in the closure of previous elements.



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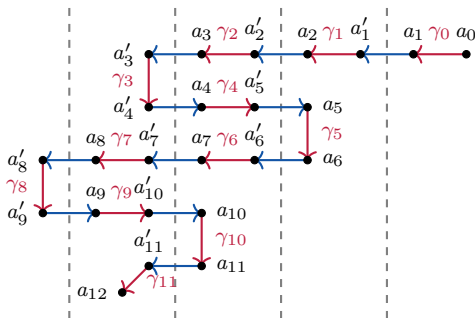
- a_{n+1} is in the closure of previous elements
 $\Rightarrow$ By modularity+IH,
 a_{n+1} is in closure of previous elements of its rank;
 we already know where it has to move
 (with IH guaranteeing independence of columns).
- a_{n+1} is not in the closure of previous elements.



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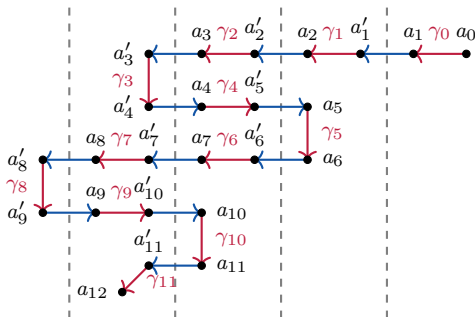
- a_{n+1} is in the closure of previous elements;
- a_{n+1} is not in the closure of previous elements
 $\Rightarrow$ By higher Neumann+pregeometry, move a_{n+1} and all of its translates to guarantee independence of columns.



Why these geometric conditions?

Induction splits into two cases:

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► Go back

The microsupported proof

Proof.

Let $w := \gamma_n x_{l_n}^{\epsilon_n} \gamma_{n-1} x_{l_{n-1}}^{\epsilon_{n-1}} \dots \gamma_1 x_{l_1}^{\epsilon_1} \gamma_0$.

$u := w$ after forgetting constants.

Let $\xi_k := \gamma_k \gamma_{k-1} \dots \gamma_0$; $\Sigma := \{1, \xi_0, \dots, \xi_n\}$.

By microsupport: there is $V \subseteq \Omega$ open such that any two members of Σ either agree on V or have disjoint images.

$G_{\Omega \setminus V} \curvearrowright V$ is also microsupported. By lawlessness, there are $h_1, \dots, h_r \in G_{\Omega \setminus V}$ such that $u(h_1, \dots, h_r) \neq 1$.

Set $p_i := \prod_{j \in D} \xi_j h_i \xi_j^{-1}$ (D selects one index per disjoint image).

👁 For $a \in V$, $w(p_1, \dots, p_r)(a) = \xi_n u(h_1, \dots, h_r)(a)$.

By construction of V (and $1, \xi_n \in \Sigma$):

- (EITHER) $\xi_n(V) \cap V = \emptyset$: $w(p_1, \dots, p_r)(V) \cap V = \emptyset$;
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