Taking model-complete cores

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Outline

Model companions

$T, S :=$first-order theories;

Definition (Model complete, model companion)

T is **model complete** if every formula is equivalent (modulo T) to an existential formula.

S is the **model companion** of T if

- S is model complete;
- S and T have the same universal consequences;

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*I.e., every model of T embeds into a model of S and vice-versa.

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If T has a model companion, this is unique;

Theorem (Saracino 1973)

Let T be an ω-categorical theory. Then, T has a model companion.*

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*ω-categorical:= a unique countable model up to isomorphism.

Core companions

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Theorem (Bodirsky 2007)

Let T be an ω-categorical theory. Then, T has a core companion.[†]

[†]See Bodirsky, Hils, and Martin 2012 and Barto, Kompatscher, Olšák, Van Pham, and Pinsker 2017 for alternative proofs.

Examples of core companions

Fact

T and S ω-categorical with countable models $\mathbb{A}$ and $\mathbb{B}$ (resp.). TFAE:

- *S is the core companion of T;*
- *$\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent*[‡] *and*
$$\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B}).^{\S}$$

Examples

- The core companion of $(\mathbb{Q}; \neq)$ is
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- The core companion of the Random graph $(R; E)$ is

[‡] There are homomorphisms $\phi : \mathbb{A} \to \mathbb{B}$ and $\psi : \mathbb{B} \to \mathbb{A}$.

[§] For every $f \in \mathrm{End}(\mathbb{B})$ and finite $C \subseteq B$, there is $g \in \mathrm{Aut}(\mathbb{B})$ such that $f_{\restriction C} = g_{\restriction C}$.

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The core companion of $\mathbb{A}$ is often "simpler" than $\mathbb{A}$.

Motivation: infinite domain CSPs

$\tau :=$ finite relational language.
$\mathbb{B} :=$ a fixed τ-structure.

Definition ($\mathrm{CSP}(\mathbb{B})$)

$\mathrm{CSP}(\mathbb{B})$ is the following computational problem:

- **INPUT:** A **finite** τ-structure $\mathbb{A}$;
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- Interested in CSPs of "nice" classes of ω-categorical structures[‡];
- "algebraic approach" to study the complexity of CSPs only works for model-complete cores;
- For $\mathbb{A}$ ω-categorical, $\mathrm{CSP}(\mathbb{A}) = \mathrm{CSP}(\mathbb{B})$, where $\mathbb{B}$ is the core companion of $\mathbb{A}$.

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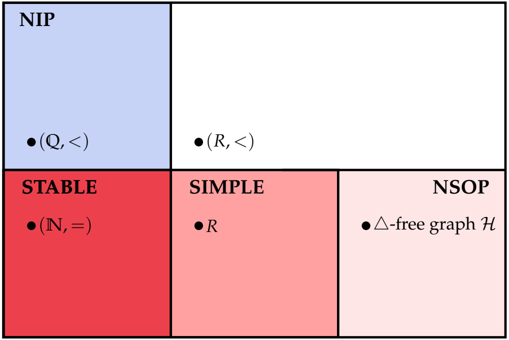
Question

What properties are preserved under taking core companions?

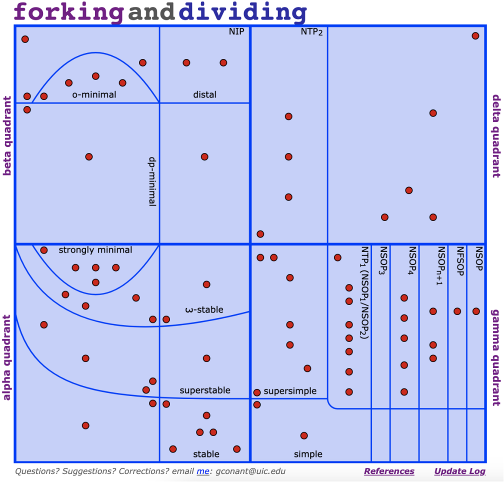
Model theoretic properties

We are interested in model theoretic properties



Model theoretic properties

But there are lots of them...



Model theoretic properties

As an example, we pick our favorite:

Definition (Stability)

$\phi(\bar{x};\bar{y})$ has the **order property** OP if there are $(\bar{a}_i\bar{b}_i)_{i\in\mathbb{N}}$ s.t.

$$\vDash \phi(\bar{a}_i;\bar{b}_j) \text{ if and only if } i<j \ .$$

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Trace definability

Definition (Trace definability, Walsberg 2021)

A **trace definition** of $\mathbb{B}$ in $\mathbb{A}$ is a map $\mathfrak{t}\colon \mathbb{B} \to \mathbb{A}^m$ such that for any[§] $\phi(\overline{x})$, there is $\psi(\overline{y})$ such that for any $\overline{a} \in \mathbb{B}^n$,

$$\mathbb{B} \vDash \phi(\overline{a}) \Leftrightarrow \mathbb{A} \vDash \psi(\mathfrak{t}(\overline{a})).$$

T **trace defines** S if some $\mathbb{B} \models S$ is trace definable in some $\mathbb{A} \models T$.

[§] We allow parameters in ϕ and ψ.

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- T is stable $\Leftrightarrow$ it does not trace define $(\mathbb{Q}; <)$;
- T is NIP $\Leftrightarrow$ it does not trace define the Random graph R;
- If T trace defines R, it trace defines every binary structure (so, simple, NSOP, etc. are not described by trace definitions).

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Theorem (Walsberg 2021)

Suppose that T trace defines S. If T has one of the following properties, then S also does:
stability, NIP, NIP_k, being totally transcendental, superstability, strong dependence, having finite rank for $\{U$, Morley, dp, $\mathrm{op}\}$-rank.

Patterned properties

Definition (Patterned property, Shelah 2000[§])

Take $\mathcal{C}, \mathcal{I} \subseteq (\mathcal{P}(n) \times \mathcal{P}(n)) \setminus \{(\emptyset, \emptyset)\}$.
$\mathcal{C}$: consistency conditions, $\mathcal{I}$: inconsistency conditions.
Call $(\mathcal{C}, \mathcal{I})$ an n-**pattern**.

$\phi(\bar{x}; \bar{y})$ **exhibits** $(\mathcal{C}, \mathcal{I})$ if there are $(\bar{b}_i)_{i<n}$ such that

1. for all $A = (A^+, A^-) \in \mathcal{C}$,
$$\{\phi(\bar{x}; \bar{b}_i) \mid i \in A^+\} \cup \{\neg\phi(\bar{x}; \bar{b}_j) \mid j \in A^-\}$$
is consistent; and
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A **patterned property**, is a set $\mathcal{P} = \{(\mathcal{C}_n, \mathcal{I}_n) \mid n \in \mathbb{N}\}$, where $(\mathcal{C}_n, \mathcal{I}_n)$ is an $f(n)$-pattern, for $f : \mathbb{N} \to \mathbb{N}$.

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A **patterned property**, is a set $\mathcal{P} = \{(\mathcal{C}_n, \mathcal{I}_n) \mid n \in \mathbb{N}\}$, where $(\mathcal{C}_n, \mathcal{I}_n)$ is an $f(n)$-pattern, for $f \colon \mathbb{N} \to \mathbb{N}$.

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$\phi(\bar{x};\bar{y})$ has OP $\Leftrightarrow$ if $\phi(\bar{x};\bar{y})$ exhibits $\mathcal{OP} := \{(\mathcal{C}_n, \mathcal{I}_n) \mid n \in \mathbb{N}\}$ for

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Fact (Bailetti 2024)

For each $\mathrm{XP} \in \{\mathrm{OP}, \mathrm{IP}, \mathrm{SOP}, k\text{-}\mathrm{TP}, k\text{-}\mathrm{TP}_2, \mathrm{SOP}_1, \mathrm{SOP}_2, \mathrm{SOP}_3\}$, *there is a patterned property* $\mathcal{XP}$ *such that*

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Preservation theorems

Theorem (BBM 2025)

Let S be the core companion of T. Then, T trace defines S.

So all properties preserved by trace definitions are preserved by taking the core companion.

Theorem (BBM 2025)

Let S be the core companion of T. Let $\mathcal{P}$ be a patterned property. If T has $N\mathcal{P}$, then so does S.

Proof idea.

Proof uses positive model theory! (Kamsma 2025).

Define the positivisation $\mathcal{P}^+$ of $\mathcal{P}$ so that:

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Let S be the core companion of T. If T has one of the following, so does S:

- *stability* ■■■;
- NIP ■■■;
- NSOP ■
- *simplicity* ■;
- NTP_2 ■;
- NSOP_n *for fixed* $n \in \mathbb{N}$ ■■;
- λ*-stability for fixed* λ ■■■;
- *monadic stability* ■;
- *monadic* NIP ■;
- *totally transcendentality* ■■■;
- *strong minimality* ■■;
- NIP_k ■;
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- *finite* U*-rank* ■.

■ type-counting argument;
■ patterned property argument;
■ trace definability;
■ patterned property variant.

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- $\mathcal{P}$ preserved by core companion $\Rightarrow$ $\mathcal{P}$ preserved by model companion.
- Was it known that model companion preserves simplicity, $\mathrm{NSOP}\ldots$?

Interpretations

Definition (Interpretation)

An **interpretation** of $\mathbb{B}$ in $\mathbb{A}$ is a partial surjection $I : \mathbb{A}^d \to \mathbb{B}$ s.t. for every atomic relation R of $\mathbb{B}^n$ (without parameters), $I^{-1}(R)$ is definable in $\mathbb{A}$ (without parameters).

Examples

- $(\mathbb{C}; 0, 1, +, \times)$ is interpretable in $(\mathbb{R}; 0, 1, +, \times)$;
- The **Johnson graph** $\mathfrak{J}(2)$
  - Domain: $[\mathbb{Q}]^2$ (unordered pairs);
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    $E := \{(\{a,b\},\{c,d\}) \in ([\mathbb{Q}]^2)^2 | \; |\{a,b\} \cap \{c,d\}| = 1\}$;
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Core interpretations

$\mathcal{C}$:= a class of ω-categorical structures;
$\mathrm{I}(\mathcal{C})$:= structures interpretable in $\mathcal{C}$;
$\mathrm{M}(\mathcal{C})$:= core companions of structures in $\mathcal{C}$.

Theorem (BBM 2025)

$$\mathrm{I}(\mathrm{MI}(\mathcal{C})) = \mathrm{MI}(\mathcal{C}).$$

$\mathbb{B}$ is **core interpretable** in $\mathbb{A}$ if $\mathbb{B} \in \mathrm{MI}(\mathbb{A})$.
We have

$$\mathrm{I}(\mathcal{C}) \subseteq \mathrm{MI}(\mathcal{C}) \subseteq \mathrm{T}(\mathcal{C}) \ ,$$

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- We know: model theoretic properties preserved by interpretations are preserved by core interpretations;
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Non-preservation of $\mathrm{I}(\mathbb{Q};=)$

Theorem (BBM 2025)

$$\mathrm{I}(\mathbb{Q};=) \subsetneq \mathrm{MI}(\mathbb{Q};=).$$

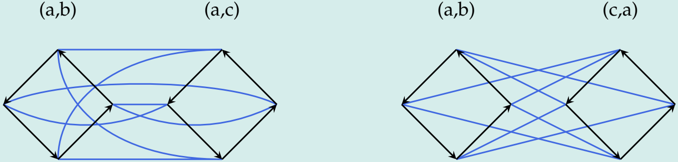
Non-preservation of $\mathrm{I}(\mathbb{Q};=)$

Proof idea.

For $\mathfrak{X} \in \mathrm{I}(\mathbb{Q};=)$, "blow-up" vertices of $\mathfrak{J}''(2)$ to 4-cycles.[§]

For (a,b) and (a,c), E connects vertices of same parity.

For (a,b) and (c,a), E connects vertices of different parity.



N holds as before.

Core companion is $\mathfrak{Y} \leq \mathfrak{X}$ on $\{(a,b,m) \in \mathbb{Q}^{(2)} \times \mathbb{Z}_4 | a < b\}$.
This is a finite cover of the Johnson graph $\mathfrak{J}(2)$.
Rest of the argument is group-theoretic.

□

[§]For the connoisseur: this is a finite cover.

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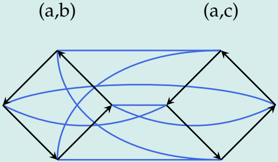

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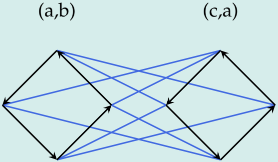

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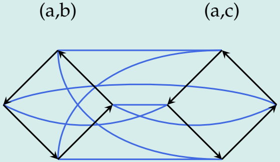

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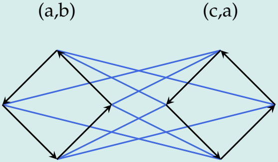

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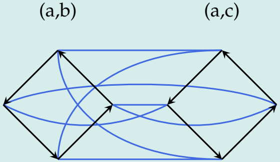

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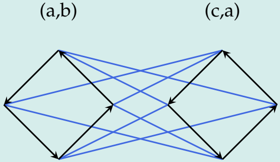

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Lachlan's class

In the 80's Lachlan studied the following class:

$\mathcal{D} := \mathbb{A}$ is ω-stable[§] and a reduct of a finitely homogeneous structure.[¶]

Lemma (BBM 2025)

$$\mathrm{MI}(\mathbb{Q};=) \subseteq \mathcal{D}.$$

[§] ω-stable:= for all finite C there are $\leq \omega$ 1-types over C.

[¶] Finitely homogeneous:= relational (in a finite language), and every automorphism between finite substructures extends to an automorphism.

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Theorem (Lachlan 1987)

$$\mathbb{A} \in \mathcal{D} \Leftrightarrow \mathbb{A} \text{ is stable and interpretable in } (\mathbb{Q}; <).$$

Lemma (BBM 2025)

$$\mathrm{MI}(\mathbb{Q}; =) \subseteq \mathcal{D}.$$

[§] ω-stable:= for all finite C there are $\leq \omega$ 1-types over C.

[¶] Finitely homogeneous:= relational (in a finite language), and every automorphism between finite substructures extends to an automorphism.

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Some conjectures

Conjecture (BBM 2025)

$$\mathrm{MI}(\mathbb{Q};=) = \mathcal{D}.$$

Our conjecture implies the weaker conjecture:

Conjecture (Walsberg 2021)

$$\mathcal{D} \subseteq \mathrm{T}(\mathbb{Q};=).$$

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Theorem (BBM 2026+)

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Proof uses Lachlan 1987 + precise description of $\mathrm{I}(\mathbb{Q}; <)$ (by EI).

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Thanks!

Recap:

- Most model theoretic properties are preserved by core companions;
- $I(\mathbb{Q};=)$ and $I(\mathbb{Q};<)$ are not preserved by core companions;
- Still a lot to understand about MI and $\mathcal{D}$.

Paper:

Tame ω-categorical world:

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