Universally measure zero non-forking formulas in simple ω-categorical Hrushovski constructions

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Outline

Invariant Keisler measures

Definition 1 (Keisler measure)

A **Keisler measure** on $\mathcal{M}$ in the variable $\overline{x}$ is a finitely additive **probability** measure on $\mathrm{Def}_{\overline{x}}(M)$:

- $\mu(X \cup Y) = \mu(X) + \mu(Y)$ for disjoint X and Y;
- $\mu\left(M^{|\overline{x}|}\right) = 1$.

We want to study Keisler measures **invariant** under automorphisms of $\mathcal{M}$:

$$\mu(X) = \mu(\sigma \cdot X) \text{ for } \sigma \in \mathrm{Aut}(M),$$

where $\sigma \cdot \phi(M^{|\overline{x}|}, \overline{a}) = \phi(M^{|\overline{x}|}, \sigma(\overline{a}))$.

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Two notions of smallness I: universally measure zero

Invariant Keisler measures yield a notion of "**smallness**":

Definition 2 (Universally measure zero, $\mathcal{O}(\emptyset)$)

A definable set $X \in \mathrm{Def}_{\overline{x}}(M)$ is **universally measure zero** if $\mu(X) = 0$ for every invariant Keisler measure.
We call $\boldsymbol{\mathcal{O}_{\overline{x}}(\emptyset)}$ the set (ideal) of definable subsets of $M^{|\overline{x}|}$ which are universally measure zero. Let $\boldsymbol{\mathcal{O}(\emptyset)}$ be the union of all these sets.

Definition 3

We say that $I \subseteq \mathrm{Def}_{\overline{x}}(M)$ is an **ideal** if:

- $\emptyset \in I$;
- If $Y \in I$ and $X \subseteq Y$, then $X \in I$; and
- If $X, Y \in I$, then $X \cup Y \in I$.

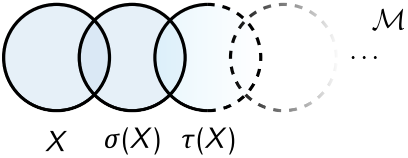
Two notions of smallness II: forking

Recall that **Forking** over ∅ is another notion of smallness for definable sets. We call ***F*(∅)** the set of definable sets forking over ∅.

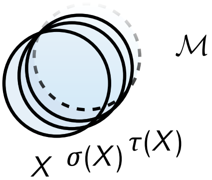
Definition 4

A formula $\phi(x, b)$ **divides** over ∅ if there is an indiscernible sequence $(b_i|i < \omega)$ with $b_0 = b$ and $k \in \omega$ such that $\{\phi(x, b_i)|i < \omega\}$ is k-inconsistent. A formula **forks** over ∅ if it is in the ideal generated by dividing formulas (in some variable).

Motto of dividing:



A **small** set can be moved around by automorphisms.



A **large** set will always overlap with itself no matter how much you try to move it.

How do $F(\emptyset)$ and $\mathcal{O}(\emptyset)$ interact?

We can compare these two ideals (in ω-saturated models of a given theory).

Theorem 5 (Folklore)

For any theory $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$.

For stable theories $F(\emptyset) = \mathcal{O}(\emptyset)$ (Chernikov et al. 2021). This should also be the case for NIP theories. It is for NIP ω-categorical theories by Braunfeld & M. (2022).

$F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ in simple theories

For **simple** structures it was unknown whether $F(\emptyset) = \mathcal{O}(\emptyset)$, until the counterexample given in

Invariant measures in simple and in small theories

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Abstract

We give examples of (i) a simple theory with a formula (with parameters) which does not fork over $\emptyset$ but has μ-measure 0 for every automorphism invariant Keisler measure μ, and (ii) a definable group G in a simple theory such that G is not definably amenable, i.e. there is no translation invariant Keisler measure on G.

What about simple ω-categorical structures?

It is natural to ask whether there are simple ω-categorical examples of $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$:

- The known example is not ω-categorical;
- In the "group analogue" of this question, there are no ω-categorical counterexamples (Chernikov et al 2021, Evans & Wagner 2000);
- An ω-categorical example would not be MS-measurable, answering negatively the following question of Elwes & Macpherson (2008):

Q: Is every ω-categorical supersimple structure MS-measurable?

Note: MS-measurable structures have a definable and finite **dimension-measure** function assigning a dimension and a measure to each definable set such that they satisfy Fubini's theorem.

▸ More on MS-measurable structures

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Ergodic measures

There is a correspondence between Keisler measures on $\mathcal{M}$ in the variable $\overline{x}$ and regular Borel probability measures on $S_{\overline{x}}(M)$.

Definition 6 (Ergodic measure)

Invariant μ is **ergodic** if for any Borel $A \subseteq S_{\overline{x}}(M)$ we have that if for any $\tau \in \mathrm{Aut}(M)$,

$$\mu(A \triangle \tau \cdot A) = 0,$$

then either $\mu(A) = 0$ or $\mu(A) = 1$.

Choosing M countable, we have an **ergodic decomposition** (Phelps 2011):

$$\mu(A) = \int_{\mathrm{Erg}_{\overline{x}}(M)} \nu(A) \mathrm{d}\mathfrak{m}(\nu).$$

Weak Algebraic Independence and Probabilistic independence

We say that $A, B \subseteq \mathcal{M}^{eq}$ are **weakly algebraically independent** (over $\emptyset$) if $\mathrm{acl}^{eq}(A) \cap \mathrm{acl}^{eq}(B) = \mathrm{acl}^{eq}(\emptyset)$. We write $A \mathop{\smile\!\!\!\!\!\!\mid}^{a} B$.

From Jahel & Tsankov (2022) we have:

Theorem 7 (Probabilistic Independence Theorem)

Let $\mathcal{M}^{eq}$ be ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$. Let μ be an ergodic measure and $a \mathop{\smile\!\!\!\!\!\!\mid}^{a} b$. Then, for any formulas $\phi(x, y), \psi(x, z)$,

$$\mu(\phi(x, a) \wedge \psi(x, b)) = \mu(\phi(x, a))\mu(\psi(x, b)).$$

Recently, Chevalier & Hrushovski (2022) have generalised these results outside of the ω-categorical context.

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Example: Random Graph

For A, B finite and disjoint subsets of the random graph R, let $\phi(x, A, B)$ be the formula saying "x is connected to all of A and none of B".

We study the ergodic measure μ and write $\mu(E(x, a)) = p$.

Disjoint sets of vertices are weakly algebraically independent, so:

$$\mu(\phi(x, A, B)) = p^{|A|}(1-p)^{|B|}.$$

Hence, by the **ergodic decomposition**:

Theorem 8 (Measures in the Random graph, Albert (1994))

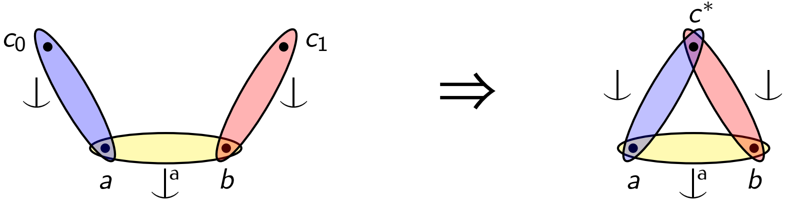
For any invariant Keisler measure $\mu : \mathrm{Def}_x(R) \to [0, 1]$*, there is a unique probability measure* $\mathfrak{m}$ *on* $[0, 1]$ *such that for any* $A, B \subseteq R$ *finite and disjoint,*

$$\mu(\phi(x, A, B)) = \int_0^1 p^{|A|}(1-p)^{|B|} \mathrm{d}\mathfrak{m}(p).$$

Strong Independence Theorem

Theorem 9 (Strong Independence Theorem)

*Let $\mathcal{M}^{eq}$ be simple, ω-categorical with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ and $F(\emptyset) = \mathcal{O}(\emptyset)$. Then, it satisfies the **strong independence theorem** (over $\emptyset$): Say $a \mathop{\smile}\limits^{|\,\mathrm{a}} b$, $c_0 \equiv c_1$ and $c_0 \mathop{\smile}\limits^{|} a$, $c_1 \mathop{\smile}\limits^{|} b$. Then, there is c^* such that $c^* \equiv_a c_0$, $c^* \equiv_b c_1$, and $c^* \mathop{\smile}\limits^{|} ab$.*



In general, simple ω-categorical structures with $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ satisfy this for $a \mathop{\smile}\limits^{|} b$. But here we have weak algebraic independence.

Proof of the Strong Independence Theorem

Let $\phi(x, a)$ and $\psi(x, b)$ isolate $\mathrm{tp}(c_0/a)$ and $\mathrm{tp}(c_1/b)$. By existence property of non-forking independence, there is $b' \equiv b$ such that $b' \mathop{\smile\!\!\!\!\!\!\mid}\, a$. By the independence theorem over $\emptyset$, $\phi(x, a) \wedge \psi(x, b')$ doesn't fork over the $\emptyset$. By $F(\emptyset) = \mathcal{O}(\emptyset)$ and the ergodic decomposition, there is an ergodic measure μ such that

$$\mu(\phi(x, a) \wedge \psi(x, b')) > 0.$$

But by the probabilistic independence theorem,

$$\begin{aligned} \mu(\phi(x, a) \wedge \psi(x, b')) &= \mu(\phi(x, a))\mu(\psi(x, b')) \\ &= \mu(\phi(x, a))\mu(\psi(x, b)) \\ &= \mu(\phi(x, a) \wedge \psi(x, b)). \end{aligned}$$

Hence, $\mu(\phi(x, a) \wedge \psi(x, b)) > 0$ and so doesn't fork over $\emptyset$.

Strategy

Q: Are there simple ω-categorical structures with $F(\emptyset) \neq \mathcal{O}(\emptyset)$?

Idea for a counterexample: A simple ω-categorical structure which does not satisfy the strong independence theorem.

Candidate: **Simple ω-categorical Hrushovski constructions**.

Why? They are the only known example of supersimple ω-categorical **not one-based** structures (i.e. weak algebraic independence $\neq$ non-forking independence). So we may be able to construct simple ones not satisfying the strong independence theorem (and indeed we are!).

My example

We build an ω-categorical supersimple Hrushovski construction $\mathcal{M}$ of SU-rank 2, which is a **graph** such that:

- $\mathrm{acl}^{eq}(\emptyset) = \mathrm{dcl}^{eq}(\emptyset)$ (by **weak elimination of imaginaries**).
- $\mathrm{Aut}(M)$ acts transitively on the vertices of M.
- There are no k-cycles for $k < 6$.
- If a, b form an edge, $a \mathop{\smile\!\!\!\!\!\!|}^{a} b$ (but not $a \mathop{\smile\!\!\!\!\!\!|} b$).
- If a and c are at distance two from each other, then $a \mathop{\smile\!\!\!\!\!\!|} c$.

By the **strong independence theorem**, if $F(\emptyset) = \mathcal{O}(\emptyset)$, $\mathcal{M}$ should contain pentagons! Hence, $F(\emptyset) \neq \mathcal{O}(\emptyset)$

Main results

Theorem 10 (Supersimple ω-categorical, $F(\emptyset) \neq \mathcal{O}(\emptyset)$)

There are supersimple ω-categorical structures with $F(\emptyset) \neq \mathcal{O}(\emptyset)$. In particular, various ω-categorical Hrushovski constructions witness this. They can be chosen to have independent n amalgamation over algebraically closed sets for arbitrarily large n (or even for all n).

Corollary 11

There are supersimple ω-categorical structures which are not MS-measurable. As above, these can be chosen to have arbitrarily strong independent n-amalgamation properties.

Remark 12

There are some previous counterexamples of the latter by Evans (2022) which also use ω-categorical Hrushovski constructions. However, Evans' counterexamples rely on not satisfying some independent n-amalgamation property.

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What about a converse?

We may ask whether satisfying the **strong independence theorem** is sufficient for $F(\emptyset) = \mathcal{O}(\emptyset)$.

I have a proof that an ω-categorical Hrushovski construction satisfying the strong independence theorem (and independent n-amalgamation for all n) is not MS-measurable. This uses a higher dimensional version of the probabilistic independence theorem in ω-categorical MS-measurable structures with independent 4-amalgamation.

Presumably, the same techniques also works for showing that $F(\emptyset) \neq \mathcal{O}(\emptyset)$.

Further Questions

- Is every ω-categorical MS-measurable structure one-based?
- Is every one-based supersimple ω-categorical structure MS-measurable?
- Is any ω-categorical supersimple not one-based Hrushovski construction such that $F(\emptyset) = \mathcal{O}(\emptyset)$ (perhaps even MS-measurable)?
- Can we classify the invariant measures on an ω-categorical Hrushovski construction?
  Hunch/conjecture: there are very few of them (e.g. only those coming from invariant types).

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MS-measurable structures

Definition 13 (Macpherson & Steinhorn, 2008)

An infinite $\mathcal{L}$-structure is **MS-measurable** if there is a **dimension measure function** $h = (\dim, \mu) : \mathrm{Def}(M) \to \mathbb{N} \times \mathbb{R}^{>0}$ such that:

Finiteness: $h(\phi(\overline{x}, \overline{a}))$ has finitely many values as $\overline{a} \in M^m$ varies;

Definability: The set of $\overline{a} \in M^m$ such that $h(\phi(\overline{x}, \overline{a}))$ has a given value is $\emptyset$-definable;

Algebraicity: For $|\phi(M^n, \overline{a})|$ finite, $h(\phi(\overline{x}, \overline{a})) = (0, |\phi(M^n, \overline{a})|)$;

Additivity: For $X, Y \subset M^n$ definable and disjoint

$$\mu(X \cup Y) = \begin{cases} \mu(X) + \mu(Y), & \text{for } \dim(X) = \dim(Y); \\ \mu(X), & \text{for } \dim(Y) < \dim(X). \end{cases}$$

Fubini for Projections: Let $X \subseteq M^n$ be definable, $\pi : M^n \to M$ be the projection on the i^{th} coordinate. Suppose for each $a \in \pi(X)$ $h(\pi^{-1}(a) \cap X) = (d, v)$. Then, $\dim(X) = \dim(\pi(X)) + d$ and $\mu(X) = \mu(\pi(X)) \times v$.

Basic facts about MS-measurable structures

Macpherson & Steinhorn (2008):

Remark 14

- Being MS-measurable is a property of a theory;
- MS-measurable structures are supersimple of finite SU-rank;
- If $\mathcal{M}$ is MS-measurable, then so is $\mathcal{M}^{eq}$.

Examples 15

- Pseudofinite fields (Chatzidakis, Van den Dries & Macintyre, 1997);
- Random Graph (Macpherson & Steinhorn, 2008);
- ω-categorical ω-stable structures, and more generally smoothly approximable structures (Elwes 2005);

MS-measurable ω-categorical structures

Theorem 16 (M. (2022))

Suppose $\mathcal{M}$ is ω-categorical and MS-measurable via a dimension-measure function $h = (d, \mu)$, then $\mathcal{M}$ is MS-measurable via a dimension-measure function $h' = (SU, \mu')$, where the dimension part is given by SU-rank.

Corollary 17

Suppose that $\mathcal{M}$ is MS-measurable and ω-categorical. Then, $F(\emptyset) = \mathcal{O}(\emptyset)$.

Go back to main presentation

ω-categorical Hrushovski constructions

We work on graphs. For A finite, we define its **predimension** to be

$$\delta(A) = \alpha|A| - |E(A)|.$$

For some f slow-growing enough, we let

$$\mathcal{K}_f := \{A \text{ finite graph } : \delta(A') \geq f(|A'|) \text{ for all } A' \subseteq A\}.$$

We can build an ω-categorical structure $\mathcal{M}_f$ as a generalisation of a Fraïssé limit, where the embeddings are given by:

$$A \leq B \text{ if } \delta(A) < \delta(B') \text{ for any finite } B' \text{ such that } A \subsetneq B' \subseteq B.$$

The **algebraic closure** of $A \subseteq M_f$ is the smallest $B \supseteq A$ such that $B \leq M_f$. And the **dimension** given by $\mathrm{d}(A) = \delta(\mathrm{acl}(A))$ naturally induces SU-rank and the notion of independence corresponding to non-forking independence.

Basically, f bounds the size of the algebraic closures and we have a lot of control on which graphs to include/exclude, provided that we need $\mathcal{K}_f$ to have the amalgamation property and to be closed under certain **independence theorem diagrams** to have simplicity.