On topological reconstruction for monoids of elementary embeddings

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Outline

Topology

$\Omega :=$ a countably infinite set.
Ω^Ω is the **full monoid** on Ω (wrt composition of functions).
Ω^Ω is endowed with the **pointwise convergence topology** τ_{pw}:

$$\text{for } a, b \in \Omega,\ \mathcal{U}_{(a,b)} := \{f \in \Omega^\Omega \mid f(a) = b\}\ ,$$

form a sub-basis of open sets for τ_{pw}.

Ω^Ω with τ_{pw} is a **Polish topological semigroup**:

- composition is continuous;
- **Polish**:=separable and completely metrisable.

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Topology and structures

We have the correspondences:

$$\left\{ \begin{array}{c} \text{closed subgroups} \\ \text{of } S_\Omega \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{automorphism groups of} \\ \text{structures with domain } \Omega \end{array} \right\}.$$

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Spaces of symmetries

$\mathbb{A}$:= a structure with domain Ω.

We can associate with $\mathbb{A}$ various spaces of symmetries:

- its **automorphism group** $\mathrm{Aut}(\mathbb{A})$;
- its **monoid of elementary embeddings** $\mathrm{EEmb}(\mathbb{A})$ (maps $\phi : \mathbb{A} \to \mathbb{A}$ preserving arbitrary first-order formulas);
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Two (still imprecise) questions on reconstruction

Question 1 (reconstructing $\mathbb{A}$?)

What information can we recover about the original structure $\mathbb{A}$ from a given space of symmetries (as a topological group/monoid)?

Question 2 (reconstructing topology?)

To what extent does the algebraic structure of a space of symmetries determine its topological structure?

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ω-categorical structures

Definition 1

$G \curvearrowright \Omega$ is **oligomorphic** if $G \curvearrowright \Omega^n$ has finitely many orbits for each $n \in \mathbb{N}$ in its diagonal action $g \circ (a_1, \ldots, a_n) = (ga_1, \ldots, ga_n)$.
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Examples 2

- $(\mathbb{N}, =)$;
- $(\mathbb{Q}, <)$;
- the random graph;
- countably infinite vector spaces over finite fields.

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If $G \curvearrowright \Omega$ is oligomorphic, acl is **locally finite**: for B finite, $\mathrm{acl}(B)$ is finite.
G has **no algebraicity** if for all $B \subseteq \Omega$ finite $\mathrm{acl}(B) = B$.
$(\mathbb{N}, =), (\mathbb{Q}, <)$, and the random graph have no algebraicity.
If G has no algebraicity, then $Z(G) = 1$, where

$$Z(G) := \{g \in G | \ \forall h \in G, hg = gh\} .$$

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Saturated structures are such that $\mathrm{EEmb}(\mathbb{A}) = \overline{\mathrm{Aut}(\mathbb{A})}$,
where $\overline{G}$ is the closure of G in Ω^Ω (with τ_{pw}).

Topology and bi-interpretations

Definition 2

An **interpretation** of $\mathbb{B}$ in $\mathbb{A}$ is a partial surjection $I : A^d \to B$ for some $d \in \mathbb{N}$ such that for each relation R of $\mathbb{B}$ defined by an atomic formula, $I^{-1}(R)$ is definable in $\mathbb{A}$ (without parameters).

Interpretations compose naturally.
If I is an interpretation of $\mathbb{B}$ in $\mathbb{A}$ and J is an interpretation if $\mathbb{A}$ in $\mathbb{B}$, we say that $\mathbb{A}$ and $\mathbb{B}$ are **bi-interpretable** if the induced partial functions $I \circ J$ and $J \circ I$ are definable in $\mathbb{B}$ and $\mathbb{A}$ respectively.

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Theorem 2 (Coquand in Ahlbrandt and Ziegler 1986, cf. Lascar 1991)

Let $\mathbb{A}$ and $\mathbb{B}$ be ω-categorical structures. Then, TFAE:

- *$\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable;*
- *$\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are isomorphic as topological groups;*
- *$\mathrm{EEmb}(\mathbb{A})$ and $\mathrm{EEmb}(\mathbb{B})$ are isomorphic as topological semigroups.*

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Automatic homeomorphicity

Definition 3

Let $\mathcal{S}$ be a topological semigroup.
$\mathcal{S}$ has **automatic homeomorphicity** (AH) if every semigroup isomorphism between $\mathcal{S}$ and a closed submonoid of Ω^Ω is a homeomorphism.

The same definition makes sense for topological groups with respect to closed subgroups of S_Ω.

Automatic homeomorphicity for automorphism groups

There are well-established methods to prove automatic homeomorphicity for $G := \mathrm{Aut}(\mathbb{A})$:

- Show that G has the **small index property** (SIP): every subgroup of index $\leq \aleph_0$ is open.
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Examples 4

- $(\mathbb{N}, =)$ (Dixon, Neumann, and Thomas 1986);
- ω-categorical ω-stable structures, and the random graph (Hodges, Hodkinson, Lascar, and Shelah 1993);
- $(\mathbb{Q}, <)$ and the countable atomless Boolean algebra (Truss 1989).

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- the random graph and the generic poset (Rubin 1994);
- generic K_n-free graphs, and Cherlin-Hrushovski example without the SIP (Barbina and Macpherson 2007);

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Can we lift automatic homeomorphicity from G to $\overline{G}$?

Lemma 5 (Bodirsky, Pinsker, and Pongrácz 2017, Lemma 12)

Let G be a closed subgroup of S_Ω with automatic homeomorphicity. Suppose:

$(\star)$ *the only injective $\Phi \in \mathrm{End}(\overline{G})$ that fixes G pointwise is $\mathrm{Id}_{\overline{G}}$.*

Then $\overline{G}$ has automatic homeomorphicity.

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When does $(\star)$ happen?

- (M. and Pinsker 2025): ALWAYS
  (as long as G is the automorphism group of a countable saturated structure)

Proving lifting

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The following can be extracted from (Pech and Pech 2018):

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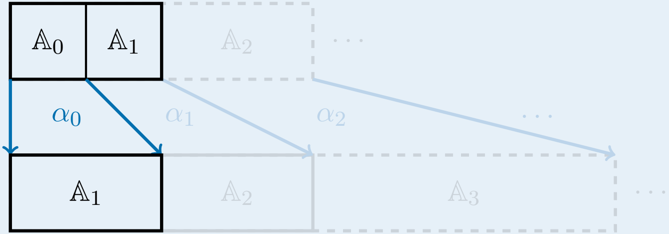
Note that $(\star)$ follows from the above.

Proving the main Lemma

Proof of Lemma.

Let $\mathbb{A}_1$ be saturated with automorphism group G and $\mathbb{A}_0 := h(\mathbb{A}_1)$ with $\alpha_0 := h^{-1}$. We can construct $\mathbb{A}_2 \succeq \mathbb{A}_1$ and an isomorphism $\alpha_1 : \mathbb{A}_1 \to \mathbb{A}_2$ extending α_0. Repeat this process to get an elementary chain $\mathbb{A}_0 \preceq \mathbb{A}_1 \preceq \mathbb{A}_2 \preceq \cdots$ with isomorphisms $\alpha_i : \mathbb{A}_{i+1} \to \mathbb{A}_i$.



Take $\mathbb{A} := \bigcup_{i<\omega} \mathbb{A}_i$, $\alpha := \bigcup_{i<\omega} \alpha_i$, and let f be an isomorphism $f : \mathbb{A} \to \mathbb{A}_1$ (exists by saturation).

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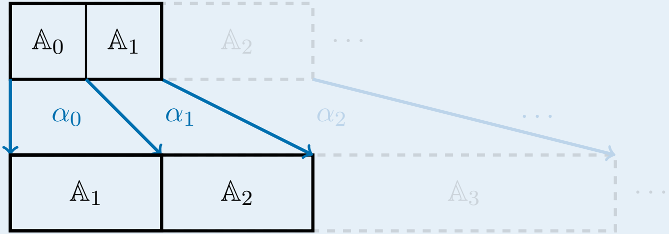

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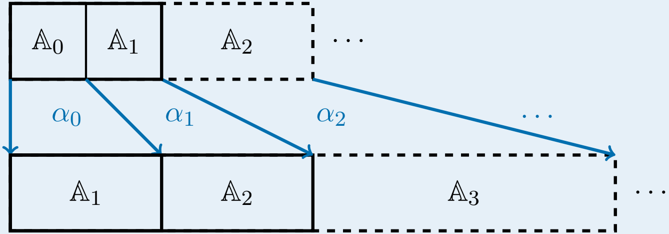

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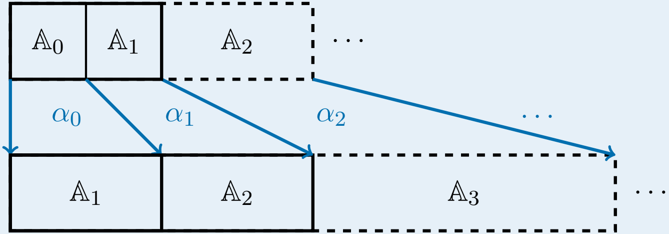

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Let $\mathbb{A}_1$ be saturated with automorphism group G and $\mathbb{A}_0 := h(\mathbb{A}_1)$ with $\alpha_0 := h^{-1}$. We can construct $\mathbb{A}_2 \succeq \mathbb{A}_1$ and an isomorphism $\alpha_1 : \mathbb{A}_1 \to \mathbb{A}_2$ extending α_0. Repeat this process to get an elementary chain $\mathbb{A}_0 \preceq \mathbb{A}_1 \preceq \mathbb{A}_2 \preceq \cdots$ with isomorphisms $\alpha_i : \mathbb{A}_{i+1} \to \mathbb{A}_i$.

Take $\mathbb{A} := \bigcup_{i<\omega} \mathbb{A}_i$, $\alpha := \bigcup_{i<\omega} \alpha_i$, and let f be an isomorphism $f : \mathbb{A} \to \mathbb{A}_1$ (exists by saturation). Let $h' := f^{-1}hf \in \mathrm{EEmb}(\mathbb{A})$. We can prove that $(\mathbb{A}_1, h) \cong (\mathbb{A}, h')$.

Finally, $f^{-1}\alpha f h' := \beta \in \mathrm{Aut}(\mathbb{A})$, meaning that

$$\alpha \circ f \circ h' = f \circ \beta,$$

as desired.

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Transfer theorem

Theorem 8 (M. and Pinsker 2025)

Let G be the automorphism group of a countable saturated structure. Suppose that G has automatic homeomorphicity (wrt closed subgroups of S_Ω). Then, $\overline{G}$ has automatic homeomorphicity (wrt closed submonoids of Ω^Ω).

Polish topologies on $\overline{G}$

For automorphism groups G, one can often prove that τ_{pw} is the unique Polish group topology on G.

Indeed, it is consistent with ZF that every Polish group has a unique Polish group topology (Solovay 1970; Shelah 1984).

Proposition 9 (Elliott, Jonušas, Mitchell, Peresse, and Pinsker 2023)

Let $\mathcal{S} \subseteq \Omega^{\Omega}$ be a monoid of injective functions whose group G of invertible elements is not closed in Ω^{Ω}. Then, there are ≥ 2 Polish semigroup topologies on $\mathcal{S}$.

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- This is the case for $\overline{G}$ where G is the automorphism group of a countable saturated structure;
- (M. and Pinsker 2025) if algebraic closure is locally finite and satisfies exchange, then there are $\geq \aleph_0$ Polish semigroup topologies on $\overline{G}$. This covers:
  - ω-categorical structures with no algebraicity;
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- All of these topologies are *finer* than τ_{pw}.

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Minimality and the Zariski topology

What about Polish topologies *coarser* than τ_{pw}?

Often it is possible to show τ_{pw} is the coarsest Hausdorff semigroup topology on $\overline{G}$ by showing it coincides with the **Zariski topology**:

Definition 10 (the Zariski topology)

Let $\mathcal{S}$ be a semigroup. The (semigroup) **Zariski** topology τ_{Z} has a sub-basis of open sets given by solutions to semigroup inequalities:

$$\{s \in \mathcal{S} \mid t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{k-1} s \dots q_1 s q_0\} ,$$

for $k, l \geq 1$ and $t_0, \dots, t_k, q_0, \dots q_l \in \mathcal{S}$.

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Let G be the automorphism group of an ω-categorical structure with no algebraicity. Then $\tau_{\mathrm{pw}} = \tau_{\mathrm{Z}}$ on $\overline{G}$.

Hence, τ_{pw} is minimal amongst Hausdorff semigroup topologies on $\overline{G}$.

Analogous statements for groups are usually false:

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When is $\tau_Z \subsetneq \tau_{pw}$?

Pinsker and Schindler 2023 give an example of a structure for which $\tau_Z \subsetneq \tau_{pw}$ on its endomorphism monoid answering a question of (Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023).

We prove the following more general statement:

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Let $G \curvearrowright \Omega$ have locally finite algebraic closure and non-trivial centre. Then, the semigroup Zariski topology τ_Z on $\overline{G}$ is not Hausdorff.

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Rubin's dream

Definition 12 (Automatic action reconstruction)

Let $\mathbb{A}$ be an ω-categorical structure with no algebraicity.
$\mathrm{Aut}(\mathbb{A})$ has **automatic action reconstruction** (AAR) if whenever $\mathbb{B}$ is another ω-categorical structure with no algebraicity and $\mathrm{Aut}(\mathbb{A}) \cong \mathrm{Aut}(\mathbb{B})$ (as groups), then $\mathbb{A}$ and $\mathbb{B}$ are bi-definable.
Define analogously AAR for $\mathrm{EEmb}(\mathbb{A})$.

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- need no algebraicity for the definition to make sense.
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We get the following surprising result:

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Theorem 13 (M. and Pinsker 2025)

Let $\mathbb{A}$ be an ω-categorical structure with no algebraicity. Then $\mathrm{EEmb}(\mathbb{A})$ has automatic action reconstruction.

This is just a consequence of

- $\tau_{\mathrm{Z}} = \tau_{\mathrm{pw}}$ (Pinsker and Schindler 2023);
- bi-interpretation between ω-categorical structures with no algebraicity yield bi-definitions (Rubin 1994; Feller and Pinsker 2025).

Thank you!

A brief recap:

For $\mathbb{A}$ an ω-categorical structure:

- **sometimes we can transfer results from $\mathrm{Aut}(\mathbb{A})$ to** $\mathrm{EEmb}(\mathbb{A})$ (automatic homeomorphicity);
- **sometimes $\mathrm{EEmb}(\mathbb{A})$ behaves much more wildly than** $\mathrm{Aut}(\mathbb{A})$ (on the number of Polish topologies);
- **sometimes $\mathrm{EEmb}(\mathbb{A})$ seems to behave better than $\mathrm{Aut}(\mathbb{A})$** (minimality of τ_{pw});
- **sometimes hard open problems for $\mathrm{Aut}(\mathbb{A})$ have "easy" answers for $\mathrm{EEmb}(\mathbb{A})$** (automatic action reconstruction).

Bibliography I

- Ahlbrandt, Gisela and Martin Ziegler (1986). "Quasi-finitely axiomatizable totally categorical theories". In: *Annals of Pure and Applied Logic* 30.1, pp. 63–82.
- Barbina, Silvia and Dugald Macpherson (2007). "Reconstruction of Homogeneous Relational Structures". In: *Journal of Symbolic Logic* 72.3, pp. 792–802.
- Bardyla, S, L. Elliott, James Mitchell, and Y Péresse (2025). "A note on intrinsic topologies of groups". In: *arXiv preprint arXiv:2506.11500*.
- Behrisch, Mike, John Truss, and Edith Vargas-García (2017). "Reconstructing the topology on monoids and polymorphism clones of the rationals". In: *Studia Logica* 105.1, pp. 65–91.
- Behrisch, Mike and Edith Vargas-García (2021). "On a stronger reconstruction notion for monoids and clones". In: *Forum Mathematicum* 33.6, pp. 1487–1506.

Bibliography II

- Bodirsky, Manuel, Michael Pinsker, and András Pongrácz (2017). "Reconstructing the Topology of Clones". In: *Transactions of the American Mathematical Society* 369, pp. 3707–3740.
- Dixon, John, Peter M. Neumann, and Simon Thomas (1986). "Subgroups of small index in infinite symmetric groups". In: *Bulletin of the London Mathematical Society* 18.6, pp. 580–586.
- Elliott, L., Julius Jonušas, Zachary Mesyan, James Mitchell, Michał Morayne, and Yann Peresse (2023). "Automatic continuity, unique Polish topologies, and Zariski topologies on monoids and clones". In: *Transactions of the American Mathematical Society* 376.11, pp. 8023–8093.
- Elliott, L., Julius Jonušas, James Mitchell, Yann Peresse, and Michael Pinsker (2023). "Polish topologies on endomorphism monoids of relational structures". In: *Advances in Mathematics* 431, p. 109214.

Bibliography III

- Feller, Roman and Michael Pinsker (2025). "Decidability of interpretability". announced.
- Ghadernezhad, Zaniar and Javier De La Nuez González (2024). "Group topologies on automorphism groups of homogeneous structures". In: *Pacific Journal of Mathematics* 327.1, pp. 83–105.
- Hodges, Wilfrid, Ian Hodkinson, Daniel Lascar, and Saharon Shelah (1993). "The Small Index Property for ω-Stable ω-Categorical Structures and for the Random Graph". In: *Journal of the London Mathematical Society* S2-48.2, pp. 204–218.
- Lascar, Daniel (1991). "Autour de la propriété du petit indice". In: *Proceedings of the London Mathematical Society* 62.1, pp. 25–53.
- Marimon, Paolo and Michael Pinsker (2025). "A guide to topological reconstruction on endomorphism monoids and polymorphism clones". to appear in the Volume in honour of Mai Gehrke of Springer's Outstanding Contributions to Logic series.

Bibliography IV

- Pech, Christian and Maja Pech (2018). "Reconstructing the Topology of the Elementary Self-embedding Monoids of Countable Saturated Structures". In: *Studia Logica* 106.3, pp. 595–613.
- Pinsker, Michael and Clemens Schindler (2023). "On the Zariski topology on endomorphism monoids of omega-categorical structures". In: *The Journal of Symbolic Logic*, pp. 1–19.
- Rubin, Matatyahu (1994). "On the reconstruction of ω-categorical structures from their automorphism groups". In: *Proceedings of the London Mathematical Society* 3.69, pp. 225–249.
- Shelah, Saharon (1984). "Can you take Solovay's inaccessible away?" In: *Israel Journal of mathematics* 48.1, pp. 1–47.
- Solovay, Robert M (1970). "A model of set-theory in which every set of reals is Lebesgue measurable". In: *Annals of Mathematics* 92.1, pp. 1–56.

Bibliography V

Truss, John (1989). "Infinite permutation groups. II. Subgroups of small index". In: *Journal of Algebra* 120.2, pp. 494–515.