Mixed identities and Neumann's Lemma

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joint work with Michael Pinsker

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1 Background
  - Equations in groups
  - Oligomorphicity
  - Mixed identities

2 On algebraic closure

3 The result

4 Strategy

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Words with constants

$G :=$ a group.

Definition (Word with constants)

A **word with constants** for G in r many variables is

$$w(x_1, \ldots, x_r) := \gamma_n \; x_{\iota_n}^{\epsilon_n} \; \gamma_{n-1} \; x_{\iota_{n-1}}^{\epsilon_{n-1}} \; \ldots \; \gamma_1 \; x_{\iota_1}^{\epsilon_1} \; \gamma_0$$

for $\gamma_0, \ldots, \gamma_n \in G$, $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$, and $\iota_1, \ldots, \iota_n \in \{1, \ldots, r\}$.

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I.e., a **term** in $\mathcal{L} := \{\cdot, ^{-1}, (\gamma)_{\gamma \in G}\}$.

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We ask that w is **reduced**: $\iota_{j-1} = \iota_j$ and $\epsilon_{j-1} = -\epsilon_j \Rightarrow\ \gamma_j \notin Z(G)$.

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w is **singular** if "forgetting constants" and reducing we get 1:

$$y^{-1} x \gamma_2 x^{-1} \gamma_1 y \xmapsto[\text{constants}]{\text{forget}} y^{-1} x x^{-1} y \mapsto y^{-1} y \mapsto 1 \; .$$

Singular words are "close" to the identity;

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Otherwise, w is **regular** (a.k.a. non-singular).

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w defines a **word map**: $w : G^r \to G$ $(g_1, \ldots, g_r) \mapsto w(g_1, \ldots, g_r)$.

- when can we **solve** w?
  $\exists g_1, \ldots, g_r \in G$ s.t. $w(g_1, \ldots, g_r) = 1$;
- when is w a **mixed identity**?
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These notions are in tension!
$\forall \overline{g} \in G\ w(\overline{g}) = 1 \Rightarrow$ for $\gamma \in G \setminus \{1\}$, $\nexists \overline{g} \in G \gamma w(\overline{g}) = 1$.

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Solving equations

Conjecture (Klyachko and Thom 2017)

G a group, $w(x_1, \dots, x_r)$ regular.
Then, there is $H \supseteq G$ s.t. $\exists h_1, \dots, h_r \in G \; w(h_1, \dots, h_r) = 1$.

- $r = 1$ this is the Kervaire-Laudenbach Conjecture:
    - Gerstenhaber and Rothaus 1962: true for all finite groups;
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Interest in finding groups for which $w(G^r)$ is large (possibly $= G$).

$$(\text{because } \gamma \notin w(G^r) \Leftrightarrow 1 \notin \gamma^{-1} w(G^r))$$

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Theorem (Adeleke and Holland 1994)

$G := \mathrm{Aut}(\mathbb{Q}; <)$. w with no constants and non-trivial.
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Then, $w(G^r) = G$.

Theorem (Mycielski 1987; Lyndon 1990)

$G := \mathrm{Sym}(\mathbb{N})$. w with no constants and $w \neq x^n$ $(n \geq 2)$.
Then, $w(G^r) = G$.

An interlude on oligomorphicity

$\mathrm{Aut}(\mathbb{Q};<)$ and $\mathrm{Sym}(\mathbb{N})$ are both oligomorphic:

Definition (Oligomorphic)

$|\Omega| = \aleph_0$. $G \curvearrowright \Omega$ is **oligomorphic** if for all $n \in \mathbb{N}$, $G \curvearrowright \Omega^n$ has finitely many orbits in the action $g \cdot (a_1, \ldots, a_n) \mapsto (ga_1, \ldots, ga_n)$.

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Many examples are **homogeneous**: every isomorphism between finite substructures of M extends to an automorphism.

Fraïssé:

$$\left\{ \begin{array}{c} \text{classes of finite substructures (ages) of} \\ \text{homogeneous structures} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Fraïssé} \\ \text{classes} \end{array} \right\}.$$

Fraïssé:* hereditary class $\mathcal{C}$ with the **amalgamation property**:

(AP) for $A, B_0, B_1 \in \mathcal{C}$, embeddings $f_i : A \to B_i$ $(i \in \{0,1\})$, there is $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ $(i \in \{0,1\})$ s.t. $g_1 \circ f_1 = g_2 \circ f_2$.

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If language is finite, M is ω-categorical.

Often, we specify homogeneous structures by their age:
random graph, generic triangle-free graph, random poset...

*Assuming language is countable and purely relational.

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$w(x_1, \ldots, x_r)$ is **a mixed identity** if $w(G^r) = 1$.

- Mixed identities describe the structure of a group/elements:

$$\forall x \; x^2 = 1 \quad \text{or} \quad \forall x \; xcx^{-1}c^{-1} = 1$$

- we would expect "rich group"⇒ "few" mixed identities.

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Theorem (Melles and Shelah 1994)

M a saturated structure[†] $\Rightarrow$ $\mathrm{Aut}(M)$ is lawless.

[†]**saturated:** types over subsets $A \subseteq M$ with $|A| < |M|$ are realised in M.

Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space (Etedadialiabadi, Gao, Le Maître, and Melleray 2021);
- transitive with free amalgamation (Ghadernezhad and de la Nuez González 2019; Bodirsky, Schneider, and Thom 2024);
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- $\mathrm{Sym}(\mathbb{N})$: $[x,t]^6$ where t is a transposition (Hull and Osin 2016);
- $\mathrm{GL}(\aleph_0, \mathbb{F}_q)$: (Bradford, Schneider, and Thom 2023);
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Mixed identities in the oligomorphic/homogeneous context

Several automorphism groups of homogeneous structures are MIF:

- random graph, generic triangle free graph, Urysohn space
  (Etedadialiabadi, Gao, Le Maître, and Melleray 2021);
- transitive with free amalgamation except if $\mathrm{Aut}(M) = \mathrm{Sym}(\mathbb{N})$
  (Ghadernezhad and de la Nuez González 2019; Bodirsky, Schneider, and Thom 2024);
- random poset, random permutation
  (Bodirsky, Schneider, and Thom 2024).

Sometimes we have mixed identities!

- $\mathrm{Sym}(\mathbb{N})$: $[x, t]^6$ where t is a transposition
  (Hull and Osin 2016);
- $\mathrm{GL}(\aleph_0, \mathbb{F}_q)$:
  (Bradford, Schneider, and Thom 2023);
- $\mathrm{Aut}(\mathbb{Q}; <)$:
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In the first two cases, we know all mixed identities are singular!

- $\mathrm{Sym}(\mathbb{N})$: all mixed identities are singular (Bodirsky, Schneider, and Thom 2024);
- $\mathrm{GL}(\aleph_0, \mathbb{F}_q)$: all mixed identities are singular (Bradford, Schneider, and Thom 2023);
- $\mathrm{Aut}(\mathbb{Q}; <)$: ???? (Bodirsky, Schneider, and Thom 2024).

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Conjecture (Bodirsky, Schneider, and Thom 2024)

$G \curvearrowright \Omega$ oligomorphic $\Rightarrow$ all mixed identities of G are singular.

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Conjecture (Bodirsky, Schneider, and Thom 2024)

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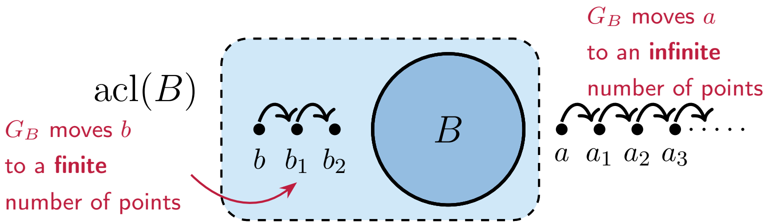
We want general conditions that imply all mixed identities are singular.

Algebraic closure

Definition (algebraic closure)

$G \curvearrowright \Omega$. $B \subseteq \Omega$ finite.

$$\mathrm{acl}(B) := \{a \in \Omega \mid a \text{ has finite } G_B\text{-orbit}\}.$$



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Lemma (Π. M. Neumann 1976)

$G \curvearrowright \Omega$. $A, B \subseteq \Omega$ finite and $A \cap \mathrm{acl}(\emptyset) = \emptyset$.
Then, there is $g \in G$ such that $(g \cdot A) \cap B = \emptyset$.

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Theorem (Abért 2005)

$G \curvearrowright \Omega$ with no algebraicity. Then, G is lawless.

Our result

Theorem (MP 2026)

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Examples (Examples from logic)

- automorphism groups of homogeneous structures whose age has the **strong amalgamation property**:
  - recovers most previous results;
  - confirms the Bodirsky, Schneider, and Thom 2024 conjecture for several cases beyond previous techniques:
    $(\mathbb{Q}; <)$ and reducts, Droste's 1985 2-homogeneous trees of negative type, homogeneous local order $S(2)$,homogeneous B, C, D relations;

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  - recovers most previous results; we also automatically cover reducts!
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    These have mixed identities (Ghadernezhad and de la Nuez González 2019);

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- Weakly branch groups (include Grigorchuk's group). Has mixed identities (Jacobson 2021);
- Any wreath product $G \wr H$ where $G \curvearrowright \Omega$ has no algebraicity. Has mixed identities (Jacobson 2021);

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- Homeomorphism groups of: Cantor set 2^ω, Hilbert cube $[0,1]^\omega$, any manifold of dimension ≥ 1;
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Proof strategy[‡]

For $g \in G$, $a_0 \in \Omega$, the images of a_0 with respect to subwords of w create a sequence $(a_0, a_1, a_1', \ldots, a_n, a_n', a_{n+1})$ with

$$g \cdot (a_1, \ldots, a_n) = (a_1', \ldots, a_n')\ .$$

[‡] w with one variable.

Proof strategy

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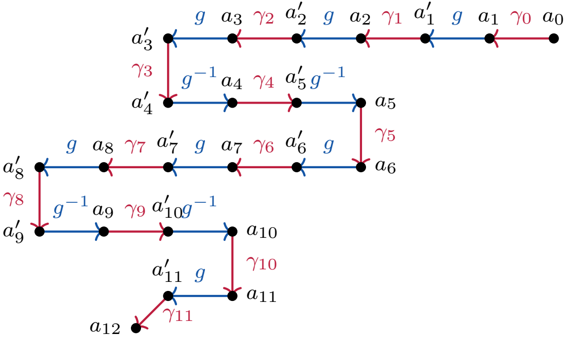
$$w(x) = \gamma_{11} x \gamma_{10} x^{-1} \gamma_9 x^{-1} \gamma_8 x \gamma_7 x \gamma_6 x \gamma_5 x^{-1} \gamma_4 x^{-1} \gamma_3 x \gamma_2 x \gamma_1 x \gamma_0$$



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For $g \in G$, $a_0 \in \Omega$, the images of a_0 with respect to subwords of w create a sequence $(a_0, a_1, a'_1, \ldots, a_n, a'_n, a_{n+1})$ with

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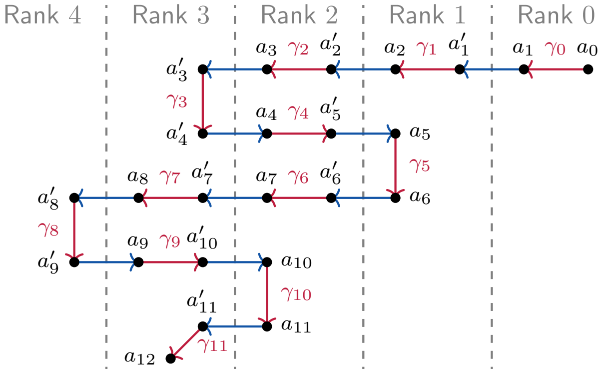
Rank elements according to their position in $\mathbb{Z}$.



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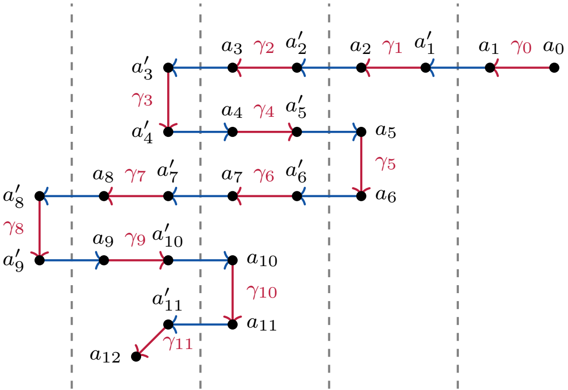
We want g so that elements of different ranks are independent!



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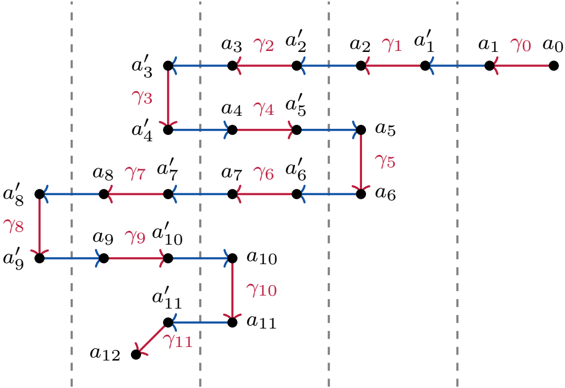
We cannot guarantee elements of same rank to be distinct!



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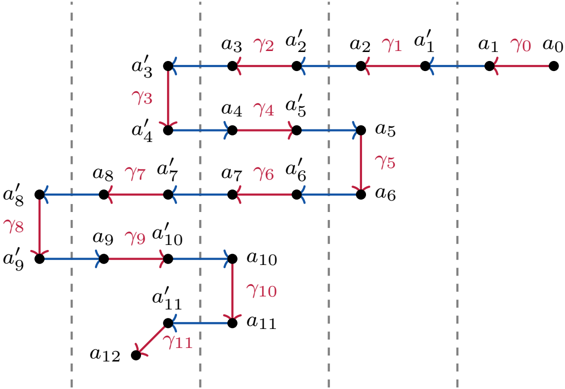
Use Neumann's Lemma to guarantee independence (looking ahead!)



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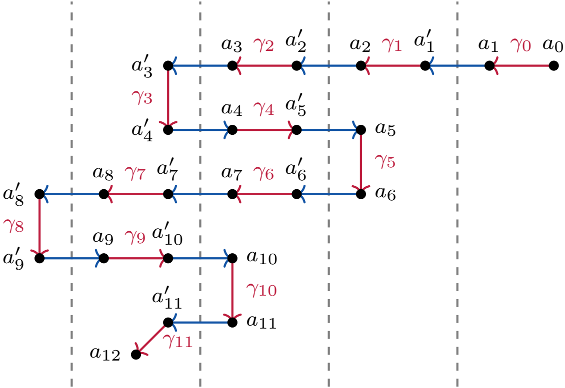
Induction proof: challenge is balancing inductive hypothesis!



The real Theorem

With algebraicity, we need a futher geometric properties and a strengthening of Neumann's Lemma.

Definition

Let $G \curvearrowright \Omega$. (Ω, acl) is **higher Neumann** if for each $B, C \subseteq \Omega$ finite, $a \in \Omega$ such that $\mathrm{acl}(a) \cap \mathrm{acl}(B) = \mathrm{acl}(\emptyset)$, and $\Sigma \subseteq G$ finite, $g \in G_B$ such that

$$\mathrm{acl}\left(\bigcup_{\gamma \in \Sigma} \gamma g(a)\right) \cap \mathrm{acl}(C) = \mathrm{acl}(\emptyset) \ .$$

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acl forms a **pregeometry** if it satisfies:

(EXCHANGE) if $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$.

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It is **modular** if for all closed $A, B \subseteq \Omega$,

$$\dim(A \cup B) = \dim(A) + \dim(B) - \dim(A \cap B) \; .$$

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Let $G \curvearrowright \Omega$. (Ω, acl) is **higher Neumann** if for each $B, C \subseteq \Omega$ finite, $a \in \Omega$ such that $\mathrm{acl}(a) \cap \mathrm{acl}(B) = \mathrm{acl}(\emptyset)$, and $\Sigma \subseteq G$ finite, $g \in G_B$ such that

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Theorem (MP 2026)

Let $G \curvearrowright \Omega$ be s.t. acl forms an infinite-dimensional higher Neumann modular pregeometry. Then all mixed identities of G are singular.

The real Theorem

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Let $G \curvearrowright \Omega$. (Ω, acl) is **higher Neumann** if for each $B, C \subseteq \Omega$ finite, $a \in \Omega$ such that $\mathrm{acl}(a) \cap \mathrm{acl}(B) = \mathrm{acl}(\emptyset)$, and $\Sigma \subseteq G$ finite, $g \in G_B$ such that

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Let $G \curvearrowright \Omega$ be s.t. acl forms an infinite-dimensional higher Neumann modular pregeometry. Then all mixed identities of G are singular.

Applies to $\mathrm{GL}(\aleph_0, \mathbb{F}_q)$ and $\mathrm{PGL}(\aleph_0, \mathbb{F}_q)$, recovering results of Bradford, Schneider, and Thom 2023.

Thank you!

Recap:

- Mixed identities are a way to describe structure in groups;
- We prove: all mixed identities are singular for groups with no algebraicity;
- Theorem covers many classes of groups for which people are studying their mixed identities.

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Extrinsic and intrinsic topologies

Our original motivation comes from studying topologies on groups/semigroups:

Definition (Topology of pointwise convergence)

$G \curvearrowright \Omega$ has a natural **pointwise convergence topology** τ_{pw} induced by the product topology on Ω^Ω (Ω discrete). Sub-basis of open sets:

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Our Theorem shows that solution sets to regular words are nowhere dense in G^r.

Definition (Zariski topologies)

The **group Zariski topology** τ_Z^G on G has as a sub-basis of open sets the solution sets to mixed disequalities in one variable:

$$\{g \in G \mid w(g) \neq 1\} \,.$$

The **semigroup zariski topology** on a semigroup S τ_Z^S is defined analogously with solution sets of semigroup disequalities:

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What can τ_{Z} describe?

- τ_{Z}^{G} is contained in any Hausdorff group topology on G;
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- they need not be Hausdorff or group/semigroup topologies!

Question

What is τ_{Z} for our favorite groups/semigroups?

We know $\tau_{\mathrm{pw}} = \tau_{\mathrm{Z}}^{G}$ for:

- $\mathrm{Sym}(\mathbb{N})$ Banakh, Guran, and Protasov 2012;
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- In Thompson's groups F and T, and $\mathrm{Homeo}(\mathbb{R})$ and $\mathrm{Homeo}(S^1)$, τ_{Z}^{G} is the **compact-open topology** τ_{co}:

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for K compact and U open. (Elliott 2026).

Zariski and MIF

Definition

A topological space X is **irreducible** if whenever $K_1, K_2 \subsetneq X$ are closed sets, we have $K_1 \cup K_2 \neq X$.

Proposition (Elliott 2026)

Let G be MIF. Then τ_Z^G is irreducible (and so not Hausdorff).

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- all semigroup mixed identities are singular!

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Corollary (de la Nuez Gonzalez, Ghadernezhad, Marimon, and Pinsker 2026)

With the above assumptions+ $Z(G) \supsetneq \{1\}$,
τ_{Z}^{S} is not Hausdorff on S.

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