Minimal and intrinsic topologies on monoids of elementary embeddings

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joint work with J. de la Nuez Gonzales, Zaniar Ghadernezhad, and Michael Pinsker

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Outline

$\mathrm{Aut}(\mathbb{A})$ and $\mathrm{EEmb}(\mathbb{A})$.

$\mathbb{A}$:= a structure.
We are interested in two natural spaces of symmetries:

- $\mathrm{Aut}(\mathbb{A})$:= automorphism group of $\mathbb{A}$;
- $\mathrm{EEmb}(\mathbb{A})$:= monoid of elementary embeddings of $\mathbb{A}$ ($f : \mathbb{A} \to \mathbb{A}$ preserving the truth of first-order formulas).

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- $\mathbb{A} := (\mathbb{N}; =)$.
  - $\mathrm{Aut}(\mathbb{A}) := \mathrm{Sym}(\mathbb{N})$;
  - $\mathrm{EEmb}(\mathbb{A}) := \mathrm{Inj}(\mathbb{N})$.
- $\mathbb{A} := (\mathbb{F}_q^{\aleph_0}; 0; +; (\lambda_r)_{r \in \mathbb{F}_q})$ (countable vector space over $\mathbb{F}_q$).
  - $\mathrm{Aut}(\mathbb{A}) := \mathrm{GL}(\aleph_0, \mathbb{F}_q)$;
  - $\mathrm{EEmb}(\mathbb{A}) :=$ injective linear transformations $T : \mathbb{F}_q^{\aleph_0} \to \mathbb{F}_q^{\aleph_0}$.

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Topologies on spaces of symmetries

We can endow $S \in \{\mathrm{Aut}(\mathbb{A}), \mathrm{EEmb}(\mathbb{A})\}$ with the **topology of pointwise convergence $\boldsymbol{\tau_{\mathrm{pw}}}$**:

$$\text{for } a, b \in \mathbb{A},\ \mathcal{U}_{a,b} := \{f \in S \mid\ f(a) = b\}\ ,$$

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When $\mathbb{A}$ is countable, with τ_{pw}:

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***Polish:** separable and completely metrizable.

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(Q1) What can we recover about $\mathbb{A}$ from S as an algebraic structure?
(Q2) What can we recover about $\mathbb{A}$ from S as an algebraic structure + τ_{pw}?
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Definition (ω-categoricity)

$|\Omega| = \aleph_0$. $G \curvearrowright \Omega$ is **oligomorphic** if for all $n \in \mathbb{N}$, $G \curvearrowright \Omega^n$ has finitely many orbits in its diagonal action
$g \circ (a_1, \ldots, a_n) = (ga_1, \ldots, ga_n)$.
$\mathbb{A}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{A})$ is oligomorphic.

Examples

- $(\mathbb{N}; =)$;
- $(\mathbb{Q}; <)$;
- the countable random graph $\mathcal{R}$;
- countably infinite vector spaces over finite fields;
- the above, with a symplectic bilinear form.

In this context, $\mathrm{EEmb}(\mathbb{A})$ is the τ_{pw}-closure of $\mathrm{Aut}(\mathbb{A})$ in $\mathbb{A}^{\mathbb{A}}$.

- So, from now on, write $G := \mathrm{Aut}(\mathbb{A})$ and $\overline{G} := \mathrm{EEmb}(\mathbb{A})$.

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Why care about topology?

Let $S \in \{\mathrm{Aut}(\mathbb{A}), \mathrm{EEmb}(\mathbb{A})\}$ for $\mathbb{A}$ ω-categorical.

- S with τ_{pw} encodes $\mathbb{A}$ up to bi-interpretability (Ahlbrandt and Ziegler 1986; Lascar 1991); ▸ More on bi-interpretations
- If we can distinguish τ_{pw} amongst the topologies compatible with the algebraic structure of S, S encodes $\mathbb{A}$ as a group/monoid!
- This gives rise to the study of **topological reconstruction**:
  - for a survey on reconstruction for $\mathrm{Aut}(\mathbb{A})$ see Macpherson 2011;
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Examples (for which τ_{pw} is minimal on $\mathrm{Aut}(\mathbb{A})$)

- $(\mathbb{N}; =)$ (Gaughan 1967);
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Examples (for which τ_{pw} is NOT minimal on $\mathrm{Aut}(\mathbb{A})$)

- $(\mathbb{Q}; <)$ (Chang and Gartside 2017): we can define a coarser **interval topology** τ_{int} with, for $a, b < c \in \mathbb{Q}$:

$$\mathcal{U}_{a,(b,c)} := \{f \in \mathrm{Aut}(\mathbb{Q}) |\ f(a) \in (b, c)\} ,$$

forming a sub-basis of open sets.

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Examples (for which τ_{pw} is NOT minimal on $\mathrm{Aut}(\mathbb{A})$)

- the **random bipartite graph** $\mathcal{B}$: we can define a coarser **localised topology** τ_{pw}^{X} (where X is one of the $\mathrm{Aut}(\mathcal{B})$-orbits) with for $a, b \in X$:

  $$\mathcal{U}_{a,b} := \{f \in \mathrm{Aut}(\mathcal{B}) \mid f(a) = b\} \ ,$$

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In a sense, the Ben Yaacov and Tsankov 2016 result is optimal:

- stable theories cannot interpret $(\mathbb{Q}; <)$;
- τ_{pw}^{X} is either not Hausdorff or τ_{pw} by **stable embeddedness**: for a G-orbit X and $a \in \mathbb{A}$, there is some finite $C \subseteq X$ such that the G_X-orbit of a equals the G_C-orbit of a.

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Question (Marimon and Pinsker 2025[†])

Are there any oligomorphic permutation groups for which τ_{pw} is not minimal on $\overline{G}$?

[†] (Pinsker and Schindler 2023).

Algebraic closure

Definition (Algebraic closure)

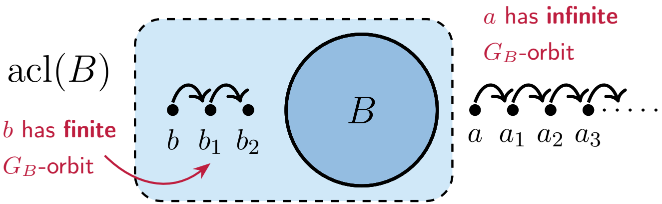
$G \curvearrowright \Omega$. For $B \subseteq \Omega$ finite,

$$\operatorname{acl}(B) := \{b \in \Omega \mid \text{the } G_B\text{-orbit of } b \text{ is finite}\}.$$

For $C \subseteq \Omega$,

$$\operatorname{acl}(C) = \bigcup_{B \subseteq C \text{ finite}} \operatorname{acl}(B).$$

$\mathbb{A}$ has **no algebraicity** if $\operatorname{acl}(B) = B$ always (for $G = \operatorname{Aut}(\mathbb{A})$).



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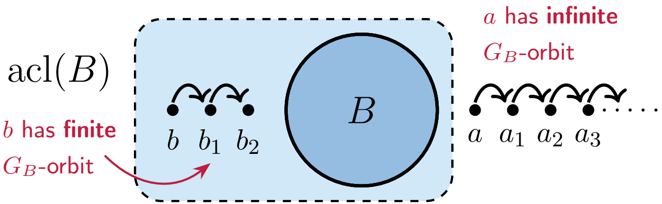

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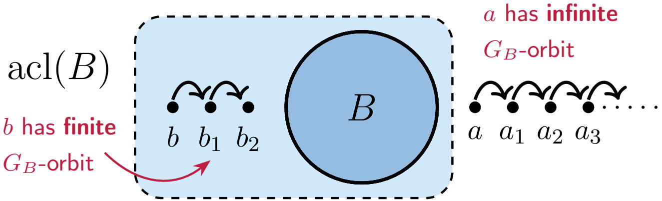

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Examples

- with no algebraicity: $(\mathbb{N}; =), (\mathbb{Q}; <)$, $\mathcal{R}$, and $\mathcal{B}$;
- with algebraicity: vector spaces, and $\aleph_0$ disjoint copies of K_2.

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For $\mathbb{A}$ ω-categorical acl is **locally finite**: for finite B, $\mathrm{acl}(B)$ is finite.

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Theorem (Pinsker and Schindler 2023[‡])

Let $\mathbb{A}$ be ω-categorical and with no algebraicity.
Then, τ_{pw} is minimal on $\mathrm{EEmb}(\mathbb{A})$.

There are predecessors in Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023 and Elliott, Jonušas, Mitchell, Peresse, and Pinsker 2023.

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$$\mathrm{acl}(C) = \bigcup_{B \subseteq C \text{ finite}} \mathrm{acl}(B).$$

$\mathbb{A}$ has **no algebraicity** if $\mathrm{acl}(B) = B$ always (for $G = \mathrm{Aut}(\mathbb{A})$).

Theorem (Pinsker and Schindler 2023)

Let $\mathbb{A}$ be ω-categorical and with no algebraicity.
Then, τ_{pw} is minimal on $\mathrm{EEmb}(\mathbb{A})$.

This is very different from the case of $\mathrm{Aut}(\mathbb{A})$!

Algebraic closure

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Theorem (Pinsker and Schindler 2023)

Let $\mathbb{A}$ be ω-categorical and with no algebraicity.
Then, τ_{pw} is minimal on $\operatorname{EEmb}(\mathbb{A})$ because it coincides with the ***semigroup Zariski topology*** *τ_{Z}.*

The Zariski topology

Definition (the Zariski topology)

$\mathcal{S}$:= a semigroup. The (semigroup) **Zariski** topology τ_Z has a sub-basis of open sets given by solutions to semigroup disequalities:

$$\{s \in \mathcal{S} \mid t_k s t_{k-1} s \ldots t_1 s t_0 \neq q_l s q_{k-1} s \ldots q_1 s q_0\},$$

for $k, l \geq 1$ and $t_0, \ldots, t_k, q_0, \ldots q_l \in \mathcal{S}$.

τ_Z is contained in any Hausdorff semigroup topology on $\mathcal{S}$. (Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023)

Analogous statements to the Pinsker and Schindler 2023 result are usually false for groups:

- semigroup Zariski on G is not Hausdorff under no algebraicity (Bardyla, Elliott, Mitchell, and Péresse 2025);
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When is $\tau_{\mathrm{Z}} \subsetneq \tau_{\mathrm{pw}}$?

Pinsker and Schindler 2023 show that for $\mathbb{A} := \aleph_0$-many copies of K_2, $\tau_{\mathrm{Z}} \subsetneq \tau_{\mathrm{pw}}$ on $\mathrm{EEmb}(\mathbb{A})$ answering a question of Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023.

We prove the following more general statement:

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We prove the following more general statement:

Theorem (de la Nuez Gonzales, Ghadernezhad, M., Pinsker 2026)

Let $G \curvearrowright \Omega$ have locally finite algebraic closure and non-trivial centre. Then, the semigroup Zariski topology τ_{Z} on $\overline{G}$ is not Hausdorff.

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The key to our result is Neumann's Lemma:

Lemma (Π.M. Neumann 1976)

$G \curvearrowright \Omega$, A, B finite, and $A \cap \mathrm{acl}(\emptyset) = \emptyset$. Then, there is $g \in G$ s.t.

$$g(A) \cap B = \emptyset \ .$$

Proof sketch

Proof sketch of the theorem.

A semigroup disequality $w(s)$

$$t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{k-1} s \dots q_1 s q_0$$

is **singular** if $k = l$ and **non-singular** otherwise.

Write $\mathcal{O}_{w(s)}$ for the τ_Z-open set of solutions to $w(s)$.

- if $w(s)$ is singular, for $\gamma \in Z(G)$,
  $1 \in \mathcal{O}_{w(s)}$ if and only if $\gamma \in \mathcal{O}_{w(s)}$;
- if $w(s)$ is non-singular, then $\overline{G} \setminus \mathcal{O}_{w(s)}$ is nowhere dense in $\overline{G}$ (wrt τ_{pw}) (use Neumann's Lemma).
- Because of this any $\gamma \in Z(G)$ cannot be separated from 1 by τ_Z-open sets (in neither G nor $\overline{G}$).

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What about minimality of τ_{pw} with algebraicity?

By our theorem, the standard method to prove minimality for τ_{pw} cannot not work in many natural ω-categorical structures with algebraicity:

- countably infinite vector spaces over finite fields (possibly with forms, or densely ordered);
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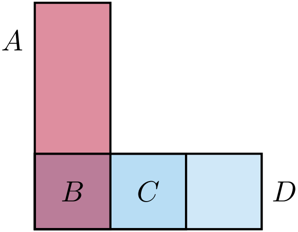
Can we still prove minimality by other means?

Base monotonicity

Definition (Base monotonicity)

$G \curvearrowright \Omega$ with locally finite acl.
acl satisfies **base monotonicity** if for finite algebraically closed $A, B \subseteq C \subseteq D$,

$$\text{if } \mathrm{acl}(A \cup B) \cap D = B, \text{ then } \mathrm{acl}(A \cup C) \cap D = C\ .$$



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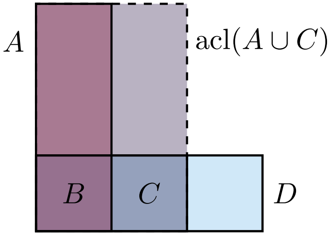

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Examples

- whenever $\mathbb{A}$ has no algebraicity, acl satisfies base monotonicity;
- vector spaces or projective spaces over finite fields (possibly with forms, or densely ordered);
- sometimes one needs to be more subtle (affine spaces over $\mathbb{F}_q$): there is $\mathbb{A}'$ with $\mathrm{EEmb}(\mathbb{A}) = \mathrm{EEmb}(\mathbb{A}')$ (with τ_{pw}) and acl satisfies base monotonicity in $\mathbb{A}'$ (but not in $\mathbb{A}$).

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Morally: acl satisfies base monotonicity when it is "not more complicated than in a vector space".

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Let $G \curvearrowright \Omega$ be oligomorphic and such that acl satisfies base monotonicity. Then, τ_{pw} is minimal on $\overline{G}$.

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Let $G \curvearrowright \Omega$ be oligomorphic and such that acl satisfies base monotonicity. Then, τ_{pw} is minimal on $\overline{G}$.

This subsumes Pinsker and Schindler 2023.

Rigidity and dispersivity

$G \curvearrowright \Omega$. Consider a semigroup topology $\tau \subseteq \tau_{\mathrm{pw}}$ on $\overline{G}$.

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$t \in \overline{G}$ is $\boldsymbol{\tau}$**-rigid** on $a \in \Omega$ if there is some τ-neighbourhood $\mathcal{V}$ of t such that, for all $s \in \mathcal{V}$, $s(a) \in \mathrm{acl}(t(a))$.

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- $t \in \overline{G}$ is τ_{pw}-rigid on Ω and not τ_{pw}-dispersive on $\mathrm{Im}(t)$.

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Lemma (de la Nuez Gonzales, Ghadernezhad, M., and Pinsker 2026)

$G \curvearrowright \Omega$ with locally finite acl. Suppose that $\tau \subsetneq \tau_{\mathrm{pw}}$ is a semigroup topology and every $t \in \overline{G}$ is τ-rigid on Ω. Then, τ is not T_1.

Proof idea of the main theorem

Proof idea.

Suppose $\tau \subsetneq \tau_{\mathrm{pw}}$ is a Hausdorff semigroup topology. Consider

$$\Delta := \{a \in \Omega | \text{ every } s \in \overline{G} \text{ is } \tau\text{-rigid on } a\} .$$

By previous Lemma, $\Delta \subsetneq \Omega$.
We can find $t \in \overline{G}$ satisfying a strong form of τ-dispersivity on $\Omega \setminus \Delta$.
By **base monotonicity**, find a "generic" copy $\mathbb{A}^\star$ of $\mathbb{A}$ within itself.
In particular,

$$\mathcal{U} := \{s \in \overline{G} | \operatorname{Im}(s) \subsetneq \mathbb{A}^\star\}$$

is τ_Z-open (and so τ-open) and a neighbourhood of t.
We exploit τ-dispersivity and genericity of $\mathbb{A}^\star$ to find:
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[‡]Usually, this copy is known as a **lovely pair** in model theory ▸ More on this.

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$$\mathcal{U} := \{s \in \overline{G} | \operatorname{Im}(s) \subsetneq \mathbb{A}^\star\}$$

is τ_Z-open (and so τ-open) and a neighbourhood of t.
We exploit τ-dispersivity and genericity of $\mathbb{A}^\star$ to find:
$u \in \mathcal{U}$ such that $\operatorname{Im}(u) \subseteq \mathbb{A}^\star$. (CONTRADICTION!)

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‡Usually, this copy is known as a **lovely pair** in model theory ▸ More on this.

Proof idea of the main theorem

Proof idea.

Suppose $\tau \subsetneq \tau_{\mathrm{pw}}$ is a Hausdorff semigroup topology. Consider

$$\Delta := \{a \in \Omega |\ \text{every } s \in \overline{G} \text{ is } \tau\text{-rigid on } a\} .$$

By previous Lemma, $\Delta \subsetneq \Omega$.
We can find $t \in \overline{G}$ satisfying a strong form of τ-dispersivity on $\Omega \setminus \Delta$.
By **base monotonicity**, find a "generic" copy $\mathbb{A}^\star$ of $\mathbb{A}$ within itself.[‡]
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Thank you!

For $\mathbb{A}$ ω-categorical

- We show τ_{Z} is not Hausdorff on $\mathrm{EEmb}(\mathbb{A})$ whenever $\mathrm{Aut}(\mathbb{A})$ has non-trivial centre;
- We find new ways to prove minimality of τ_{pw} as long as acl is sufficiently tame.

Survey:

Paper:

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Bi-interpretations

Definition

An **interpretation** of $\mathbb{B}$ in $\mathbb{A}$ is a partial surjection $I : A^d \to B$ for some $d \in \mathbb{N}$ such that for each relation R of $\mathbb{B}$ defined by an atomic formula, $I^{-1}(R)$ is definable in $\mathbb{A}$ (without parameters).

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Example

$(\mathbb{C}; 0; 1; \times; +)$ is interpretable in $(\mathbb{R}; 0; 1; \times; +)$ (by the Argand plane).

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Interpretations compose naturally.
If I is an interpretation of $\mathbb{B}$ in $\mathbb{A}$ and J is an interpretation if $\mathbb{A}$ in $\mathbb{B}$, $\mathbb{A}$ and $\mathbb{B}$ are **bi-interpretable** if $I \circ J$ and $J \circ I$ are definable in $\mathbb{B}$ and $\mathbb{A}$ respectively.

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Example

$(\mathbb{N}; =)$ is bi-interpretable with the **Johnson graph** $\mathfrak{J}(2)$:

- Domain: $[\mathbb{N}]^2$ (unordered pairs);
- Edges $E := \{(\{a, b\}, \{c, d\}) \in ([\mathbb{N}]^2)^2 | \, |\{a, b\} \cap \{c, d\}| = 1\}$.

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Theorem (Coquand in Ahlbrandt and Ziegler 1986, cf. Lascar 1991)

Let $\mathbb{A}$ and $\mathbb{B}$ be ω-categorical structures. Then, TFAE:

- $\mathbb{A}$ *and* $\mathbb{B}$ *are bi-interpretable;*
- $\mathrm{Aut}(\mathbb{A})$ *and* $\mathrm{Aut}(\mathbb{B})$ *are isomorphic as topological groups;*
- $\mathrm{EEmb}(\mathbb{A})$ *and* $\mathrm{EEmb}(\mathbb{B})$ *are isomorphic as topological monoids.*

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Lovely pairs

See Berenstein and Vassiliev 2010 (cf. Ben-Yaacov, Pillay, and Vassiliev 2003):

Definition (Lovely pair)

$\mathbb{A}$ ω-categorical. $L \subseteq \mathbb{A}$ algebraically closed is a **lovely pair** if

- (coheir) for finite algebraically closed A, B with $\mathrm{acl}(B) \cap \mathrm{acl}((L \cap B) \cup A) = \mathrm{acl}(L \cap B)$, there is $g \in G_B$ such that $g(A) \subseteq L$;
- (extension) for finite algebraically closed A, B there is $g \in G_B$ with $\mathrm{acl}(g(A) \cup B) \cap L = B$.

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Examples

- in $(\mathbb{N}; =)$ this is an infinite-coinfinite set;
- in $(\mathbb{Q}; <)$ this is a dense codense set;
- in vector space $\mathbb{F}_q^{\aleph_0}$ this is an infinite-dimensional subspace of infinite codimension.

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We need a slight variant of a lovely pair (but in all examples this is just a lovely pair);

What is left out?

- Droste's almost homogeneous semilinear orders
  (but for these we can prove $\tau_{\mathrm{pw}} = \tau_{\mathrm{Z}}$);
- Hrushovski constructions
  (but these seem to be amenable to a variant of our proof);
- atomless Boolean algebras
  (τ_{pw} should be minimal by other methods).