Exchangeability of consistent random expansions

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Outline

Randomly expanding hereditary classes

$\mathcal{L}, \mathcal{L}'$: disjoint relational languages;
$\mathcal{C}, \mathcal{C}'$: hereditary classes of finite labelled $\mathcal{L}$ and $\mathcal{L}'$ structures (resp.);

$H \star H'$ is the **free superposition** of H and H', so $H \star H' \restriction_{\mathcal{L}} = H$ and $H \star H' \restriction_{\mathcal{L}'} = H'$;

For H a finite $\mathcal{L}$-structure,

$$\mathrm{Struc}(H, \mathcal{C}') = \{H \star H' | H' \in \mathcal{C}'\}.$$

Definition 1 (consistent random expansion, $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$)

A **consistent random expansion** of $\mathcal{C}$ by $\mathcal{C}'$ assigns to each $H \in \mathcal{C}$ a probability distrubution $\mathbb{P}_H$ on $\mathrm{Struc}(H, \mathcal{C}')$ such that for $H, G \in \mathcal{C}$, $\phi : H \to G$ an embedding and $H' \in \mathcal{C}'$ such that $|H| = |H'|$,

$$\mathbb{P}_H(H \star H') = \mathbb{P}_G(\phi(H) \star H').$$

Invariant random expansions

When $\mathcal{C}$ is a Fraïssé class with Fraïssé limit $\mathcal{M}$, CREs correspond to:

Definition 2 (Invariant random expansion, $\mathrm{IRE}(M, \mathcal{C}')$)

Let $\mathcal{M}$ be a countable structure. An **invariant random expansion** of $\mathcal{M}$ by $\mathcal{C}'$ is an $\mathrm{Aut}(M)$-invariant Borel probability measure on

$$\mathrm{Struc}(M, \mathcal{C}') = \{M \star N | \ \mathrm{Age}(N) \subseteq \mathcal{C}'\}.$$

Exchangeability

Example 3 (Exchangeable structures)

We call consistent random expansions of $\mathcal{C} := \{\text{sets with no structure}\}$ **exchangeable**.

- Standard construction of the random graph is an exchangeable graph;
- Aldous 1981 and Hoover 1979 give a representation theorem for exchageable graphs and hypergraphs generalising De Finetti 1929;
- Heavily studied in probability and combinatorics[1];
- Exchangeable $\mathcal{C}'$-expansions yield consistent random $\mathcal{C}'$-expansions of $\mathcal{C}$ for any $\mathcal{C}$.

[1] See Aldous 2010, Kallenberg 1997, Janson and Diaconis 2008, and Austin 2008 for reviews.

Consistent Random Orderings

Example 4 (Consistent Random Orderings)

Consider $\mathcal{C}'$ = linear orders;

- There is a unique exchangeable ordering: for $a_1, \dots, a_k$,

$$\mathbb{P}_{a_1,\dots,a_k}(a_1 < \cdots < a_k) = \frac{1}{k!}\ ;$$

- Angel, Kechris, and Lyons 2014: this is the unique consistent random ordering for $\mathcal{C}$ one of
  - k-hypergraphs;
  - K_n^r-free hypergraphs;
  - metric spaces with rational distances.

Random Expansions of hypergraphs

Example 5

For $\mathcal{C} = \{k\text{-hypergraphs}\}$:

- Crane and Towsner 2018, and Ackerman 2021 obtain a representation theorem similar to the Aldous-Hoover theorem;
- If $\mathcal{C}'$ has all relation of arity $< k$, all $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$ are exchangeable;
- They obtain more general results under **disjoint n-amalgamation for all n** (roughly: no 'interesting' omitted substructures).

Problems on $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$

Two natural problems[2] are:

Problem 1 (What do $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$ look like?)

Given $\mathcal{C}$ and $\mathcal{C}'$, can what do the consistent random $\mathcal{C}'$-expansions of $\mathcal{C}$ look like?

Problem 2 (When do we get exchangeability?)

What conditions *prima facie* weaker than exchangeability imply exchangeability?

[2] Appearing in some form in Aldous 1985; Kallenberg 2008; Crane and Towsner 2018; Crane 2018.

A summary of previous strategies

Previous work follows one of two strategies:

(A) Choose $\mathcal{C}$ so that for lots of $\mathcal{C}'$, we can understand $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$:

- $\mathcal{C} := \{\text{sets}\}$ (De Finetti 1929; Aldous 1981; Hoover 1979);
- $\mathcal{C} := \{\text{linear orders}\}$ (Kallenberg 1997);
- $\mathcal{C} := \{k\text{-hypergraphs}\}$ (Crane and Towsner 2018; Ackerman 2021).

Only works for $\mathcal{C}$ with a unique structure in each size or no interesting omitted substructures!

(B) Choose $\mathcal{C}'$ so that for lots of $\mathcal{C}$ we can understand $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$:

- $\mathcal{C}'$ is unary (De Finetti 1929; Jahel and Tsankov 2022);
- $\mathcal{C}' := \{\text{linear orders}\}$ (Angel, Kechris, and Lyons 2014; Balister, Bollobás, and Janson 2015; Jahel and Tsankov 2022).

Only works for $\mathcal{C}'$ with very slow growth!

Our interest: consistent random graph expansions of 3-hypergraphs with some omitted configurations (e.g. K_4^3).

Main theorem

Adapting techniques from Angel, Kechris, and Lyons 2014:

Main Theorem 6 (Braunfeld, Jahel, and M. 2024)

Let $\mathcal{C}$ be k-overlap closed and let $\mathcal{C}'$ have labelled growth rate $O(e^{n^{k+\delta}})$ for every $\delta > 0$.
Then every consistent random $\mathcal{C}'$-expansion of $\mathcal{C}$ is exchangeable.

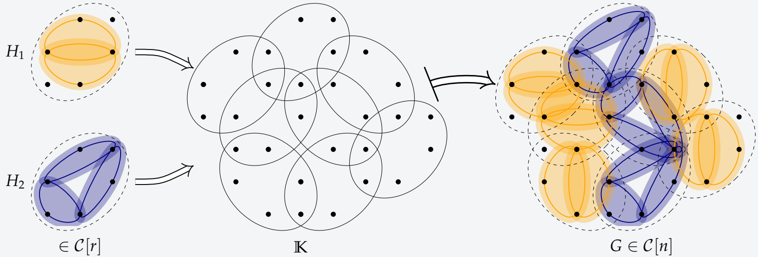
- **k-overlap closed**: $(k+1)$-hypergraphs, K_n^{k+1}-free $k+1$-hypergraphs, and many more ...
- $\boldsymbol{O(e^{n^{k+\delta}})}$: $\mathcal{C}'$ has finitely many relations of arity $\leq k$.

k-overlap closed classes

Definition 7 (k-overlap closedness)

$\mathcal{L}$ of arity $> k$. $\mathcal{C}$ is ***k*-overlap closed** if for every $r > k$ and arbitrarily large n, there exists an r-uniform hypergraph $\mathbb{K}$ on n vertices s.t.

1. $\mathbb{K}$ has at least $C(r)n^{k+\alpha(r)}$ many hyperedges for some $\alpha(r) > 0$;
2. No two $\mathbb{K}$-hyperedges intersect in more than k points;
3. For every $H_1, H_2 \in \mathcal{C}[r]$, pasting them into the $\mathbb{K}$-hyperedges yields $G \in \mathcal{C}[n]$ (possibly after adding extra relations).



Thoughts on k-overlap closedness

Main Theorem uses a random placement construction and probabilistic methods. ▸ Want to see more?

Definition 8 (k-irreducible)

A is k-irreducible if every k-many vertices from A are in some relation.

By probabilistic methods we prove k-overlap closedness for $\mathcal{C} = \mathrm{Forb}(\mathcal{F})$ with all relations of arity $> k$, where $A \in \mathcal{F}$ are:

1. $(k+1)$-irreducible; OR
2. of bounded size and k-irreducible (for $k \geq 2$).

For $k = 1$ in 1, we recover Angel, Kechris, and Lyons 2014.

Nonexamples:

- linear orders are not 1-overlap closed;
- two-graphs are not 2-overlap closed.

Consequences for invariant Keisler measures I

Definition 9 (Invariant Keisler measure)

Let $\mathcal{M}$ be countably infinite. An **invariant Keisler measure** (IKM) is an $\mathrm{Aut}(M)$-invariant Borel probability measure μ on $\mathrm{Def}_x(M)$.

- Heavily studied in model theory with several applications to Szemerédi Regularity;
- Albert 1994 and Ensley 1996 described the IKMs for homogeneous graphs and (roughly) ω-categorical NIP structures;
- IKMs for homogeneous hypergraphs are HARD! Because:
  - There are good techniques (Hruhsovski 2012; Jahel and Tsankov 2022) for
  $$\mu(\phi(x,a) \wedge \psi(x,b)),$$
  - There are very few techniques (cf. Hrushovski 2024) for
  $$\mu(\phi(x,ab) \wedge \psi(x,bc) \wedge \xi(x,ac)).$$

Consequences for invariant Keisler measures II

- For $\mathcal{M}$ homogeneous, invariant Keisler measures can be viewed as a special case of invariant random expansions;
- We describe the spaces of invariant Keisler measures for many homogeneous structures of higher arity (e.g. the universal homogeneous K_4^3-free 3-hypergraph);
- We build many (i.e. $2^{\aleph_0}$) model-theoretically tame counterexamples to conjectures on invariant Keisler measures which were recently disproven with more ad-hoc non-tame examples (Chernikov, Hrushovski, Kruckman, Krupiński, Moconja, Pillay, and Ramsey 2023; Marimon 2023; Evans 2022).

Problems for the future

Problem 3

Let $\mathcal{C}$ have free amalgamation and arity $> k$. Can we prove exchangeability of consistent random $\mathcal{C}'$-expansions, where $\mathcal{C}'$ has labelled growth rate $O(e^{n^{k+\delta}})$ for all $\delta > 0$?

Problem 4

Can we understand more systematically failures of exchangeability of $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$ when $\mathcal{C}$ and $\mathcal{C}'$ have similar growth rates?

Problem 5

Can we provide an Aldous-Hoover-like representation theorem for expansions of any arity of some of the classes we study? e.g. consistent random expansions of $\mathcal{C} = \{\text{triangle-free graphs}\}$? (cf. Crane and Towsner 2018)

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The key lemma for exchangeability

Lemma 10 (Braunfeld, Jahel, and M. 2024)

Suppose that for all $\mathbf{H}_1$, $\mathbf{H}_2 \in \mathcal{C}[k]$, *and* $\epsilon > 0$, *there is some* n, $\mathbf{G} \in \mathcal{C}[n]$ *and non-empty families* Θ_i *of embeddings of* $\mathbf{H}_i$ *in* $\mathbf{G}$ *such that for all* $\mathbf{H}' \in \mathcal{C}'[k]$ *and* $\mathbf{G}' \in \mathcal{C}'[n]$ *we have*

$$\left| \frac{N_{\Theta_1}(\mathbf{H}_1^\star, \mathbf{G}^*)}{|\Theta_1|} - \frac{N_{\Theta_2}(\mathbf{H}_2^\star, \mathbf{G}^*)}{|\Theta_2|} \right| < \varepsilon,$$

where $\mathbf{G}^\star := \mathbf{G} \star \mathbf{G}', \mathbf{H}_i^\star := \mathbf{H}_i \star \mathbf{H}'$ *and* $N_{\Theta_i}(\mathbf{H}_i^\star, \mathbf{G}^\star)$ *is the number of embeddings in* Θ_i *that are also embeddings of* $\mathbf{H}_i^\star$ *in* $\mathbf{G}^\star$.
Then every consistent random $\mathcal{C}'$*-expansion* μ *of* $\mathcal{C}$ *is exchangeable.*

Back to main presentation