When invariance implies exchangeability

Paolo Marimon

Joint work with

Samuel Braunfeld and Colin Jahel

TU Wien

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Outline

Exchangeable graphs

$\mathrm{Graph}(\mathbb{N}) :=$ space of graphs with vertex set $\mathbb{N}$.

Consider $\mathrm{Graph}(\mathbb{N})$ with the **pointwise convergence topology**: basis of clopen sets: for each finite graph $G = (V, E)$ with $V \subseteq \mathbb{N}$,

$$N_G := \{\mathcal{N} \in \mathrm{Graph}(\mathbb{N}) |\, \mathcal{N} \restriction_V = G\}.$$

Definition 1 (Exchangeable graph)

An **exchangeable graph** is a Borel probability measure μ on $\mathrm{Graph}(\mathbb{N})$ whose distribution is invariant under all permutations, i.e. $S_\infty \curvearrowright \mathbb{N}$.

Fun facts about exchangeable graphs

Standard construction of the random graph is an exchangeable graph.

Studied in:

- **Probability:**
  - Aldous 1981 and Hoover 1979 representation theorem for exchangeable graphs and hypergraphs, generalising De Finetti 1929 (characterising exchangeable colourings);
- **Combinatorics:**
  - (ergodic) exchangeable graphs correspond to graphons, the main object of the theory of graph limits (Diaconis and Janson 2008).
- **Statistical networks:**
  - Exchangeability is a natural assumption when modelling a network on a large population for which we have no information.

The Aldous-Hoover theorem

Theorem 2 (Aldous 1981 and Hoover 1979)

Let μ be an exchangeable graph.
Then, there is a Borel function[1] $f : [0,1]^4 \to \{0,1\}$ *and* $\mathrm{Uniform}[0,1]$ *independent identically distributed random variables*

$$U_\emptyset, (U_a | a \in \mathbb{N}), (U_{\{a,b\}} | \{a,b\} \in [\mathbb{N}]^2)$$

such that the random graph built by setting

$$E(a,b) \text{ if and only if } f(U_\emptyset, U_a, U_b, U_{\{a,b\}}) = 1 \qquad (\diamondsuit)$$

has the same distribution as μ.

EASY TO SEE: ($\diamondsuit$) gives an exchangeable graph.
HARD TO PROVE: any exchangeable graph is of the form ($\diamondsuit$).

[1] symmetric in the second and third argument.

What about other symmetries and other structures?

Question 1 (Aldous 1985; Kallenberg 2008; Crane and Towsner 2018)

Can we describe random graphs/hypergraphs/structures whose distribution is invariant under different symmetries?

For us: different symmetries = other permutation groups $G \curvearrowright \mathbb{N}$;

Describing can take two forms:

- A "representation theorem" like that of Aldous-Hoover;
- Showing that G-invariance implies G'-invariance for some larger group of permutations.

Invariant Random Expansions

$\mathcal{L}, \mathcal{L}'$: disjoint (finite) relational languages.

$\mathcal{C}'$: hereditary class of (labelled) finite $\mathcal{L}$-structures.
e.g. graphs, hypergraphs, trees, linear orderings;

$\mathrm{Struc}(\mathcal{C}')$: class of $\mathcal{L}'$-structures with domain $\mathbb{N}$ and **age**, i.e. class of finite substructures, $\subseteq \mathcal{C}'$.

$\mathcal{M}$: an $\mathcal{L}$-structure with domain $\mathbb{N}$.

Definition 3 (IRE)

An **invariant random expansion** of $\mathcal{M}$ by $\mathcal{C}'$, $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$, is an $\mathrm{Aut}(M)$-invariant Borel probability measure on $\mathrm{Struc}(\mathcal{C}')$.

Exchangeable structures are IREs of $(\mathbb{N}, =)$.

Homogeneous structures

We focus on $\mathcal{M}$ **homogeneous**: isomorphisms between finite substructures extend to an automorphisms of $\mathcal{M}$.

When a class of finite structures $\mathcal{C}$ forms a **Fraïssé class** we can build a countable homogeneous structure $\mathcal{M}$ whose age is $\mathcal{C}$. We call $\mathcal{M}$ the **Fraïssé limit** of $\mathcal{C}$.

Examples 4

Homogeneous structure	Fraïssé class
Random graph	graphs
$(\mathbb{Q}, <)$	linear orders
Generic tetrahedron-free 3-hypergraph	tetrahedron-free 3-hypergraphs

Why homogeneous structures? I

- Many combinatorially interesting classes of structures are Fraïssé. So IREs of homogeneous structures frequently arise naturally:
    - IREs of $(\mathbb{Q}, <)$: contractable sequences (Ryll-Nardzewski 1957) and arrays (Kallenberg 1997);
    - IREs of homogeneous unary structures: stochastic block model (Holland, Laskey, and Leinhardt 1983);
    - IREs of structures with equivalence relations have applications to spin glass models (Austin and Panchenko 2014).
- Describing IREs of arbitrary structures is essentially intractable. However, when $\mathrm{Aut}(M)$ is "very large" this becomes possible (e.g. $\mathcal{M}$ is homogeneous or ω-categorical);

Why homogeneous structures? II

For $V \subseteq \mathbb{N}$, write $\mathcal{C}'[V]$ for the structures in $\mathcal{C}'$ with domain V.

- For $\mathcal{M}$ homogeneous, $\mu \in \mathrm{IRE}(\mathcal{M}, \mathcal{C}')$, and $V \subseteq \mathbb{N}$ finite, the induced distribution of μ on $\mathcal{C}'[V]$ is determined by the isomorphism type of $\mathcal{M} \restriction_V$;
- We also study **consistent random expansions** of an hereditary class $\mathcal{C}$ by another $\mathcal{C}'$, $\mathrm{CRE}(\mathcal{C}, \mathcal{C}')$ ▸ Want to see the definition?. When $\mathcal{C}$ is Fraïssé with limit $\mathcal{M}$, these correspond to $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$.

Past progress in describing IREs takes one of two strategies:

(A) Choose $\mathcal{M}$ so that we can understand $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$ for many $\mathcal{C}'$;

(B) Choose $\mathcal{C}'$ so that we can understand $\mathrm{IRE}(\mathcal{M}, \mathcal{C}')$ for many $\mathcal{M}$.

Limitations of the strategies

Strategy (A) only works for $\mathcal{M}$ very well-behaved:

- $\mathcal{M}$ **very non-random looking:** a unique substructure (up to isomorphism) in each size:
  - $(\mathbb{Q}, <)$ (Kallenberg 1997);
  - $(\mathbb{N}, =)$ (Aldous 1981; Hoover 1979).
- $\mathcal{M}$ **very random looking:** "no interesting omitted configuration":
  - structures with disjoint n-amalgamation, e.g. random graph, random hypergraphs (Crane and Towsner 2018; Ackerman 2021).

Strategy (B) only works for $\mathcal{C}'$ with very slow growth-rate:

- $\mathcal{C}'$ is **unary** (De Finetti 1929; Jahel and Tsankov 2022);
- $\mathcal{C}' = \{$**linear orders**$\}$ (Angel, Kechris, and Lyons 2014; Balister, Bollobás, and Janson 2015; Jahel and Tsankov 2022)

What about IREs of the generic tetrahedron-free 3-hypergraph by graphs?

Exchangeability of IREs I

The IREs of $\mathcal{M}$ by $\mathcal{C}'$ always contain the exchangeable $\mathcal{C}'$-structures.

Question 2

What are (interesting) sufficient conditions for all IREs of $\mathcal{M}$ by $\mathcal{C}'$ to be exchangeable?

Reducing to the exchangeable case allows us to use the powerful theory of exchangeability.

This happens in many of the examples we mentioned:

In Strategy (A):

- For $\mathcal{M}$ with n-DAP for all n and k-transitive (i.e. all k-tuples are in the same orbit), IREs of $\mathcal{M}$ by l-hypergraphs for $l \leq k$ are exchangeable (Crane and Towsner 2018; Ackerman 2021).

Exchangeability of IREs II

In Strategy (B):

Theorem 5 (Jahel and Tsankov 2022)

Let $\mathcal{M}$ be transitive, ω-categorical, with trivial algebraicity and weak elimination of imaginaries:

- *Unary* IRE*s of $\mathcal{M}$ are exchangeable;*
- *If $\mathcal{M}$ has no $\emptyset$-definable linear ordering, then all* IRE*s of $\mathcal{M}$ by linear orders are exchangeable.*

Note: there is a unique exchangeable linear ordering μ concentrating on the isomorphism type of $(\mathbb{Q}, <)$. For all $a_1, \ldots, a_n \in \mathbb{N}$,

$$\mu(a_1 < a_2 < \cdots < a_n) = \frac{1}{n!}$$

Main theorem

Adapting techniques from Angel, Kechris, and Lyons 2014:

Main Theorem 6 (Braunfeld, Jahel, and M. 2024)

Let $\mathcal{M}$ be a homogeneous structure with k-overlap closed age.
Let $\mathcal{C}'$ have labelled growth rate $O(e^{n^{k+\delta}})$ for every $\delta > 0$.
Then every invariant random expansion of $\mathcal{M}$ by $\mathcal{C}'$ is exchangeable.

▸ Want more details on the proof?

- ***k*-overlap closed**: $(k+1)$-hypergraphs, K_n^{k+1}-free $(k+1)$-hypergraphs, and many more ...
- $\boldsymbol{O(e^{n^{k+\delta}})}$: $\mathcal{C}'$ has finitely many relations of arity $\leq k$.

Proof also works for:

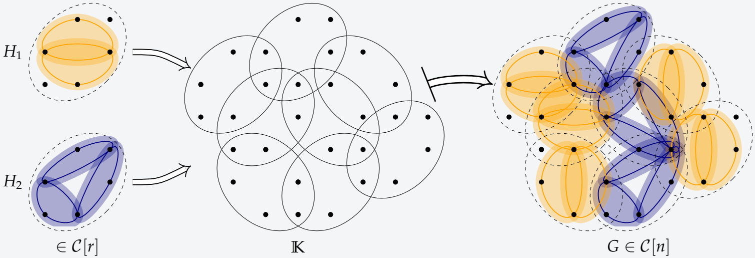
- consistent random expansions of $\mathcal{C}$ by $\mathcal{C}'$;
- reducts of $\mathcal{M}$.

k-overlap closed classes

Definition 7 (k-overlap closedness)

$\mathcal{L}$ of arity $> k$. $\mathcal{C}$ is **k-overlap closed** if for every $r > k$ and arbitrarily large n, there exists an r-uniform hypergraph $\mathbb{K}$ on n vertices s.t.

1. $\mathbb{K}$ has at least $C(r)n^{k+\alpha(r)}$ many hyperedges for some $\alpha(r) > 0$;
2. No two $\mathbb{K}$-hyperedges intersect in more than k points;
3. For every $H_1, H_2 \in \mathcal{C}[r]$, pasting them into the $\mathbb{K}$-hyperedges yields $G \in \mathcal{C}[n]$ (possibly after adding extra relations).



Thoughts on k-overlap closedness I

Definition 8 (k-irreducible)

A is k-irreducible if every k-many vertices from A are in some relation.

By probabilistic methods we prove k-overlap closedness for $\mathcal{C} = \mathrm{Forb}(\mathcal{F})$ with all relations of arity $> k$, where $A \in \mathcal{F}$ are:

1. $(k+1)$-irreducible; OR
2. of bounded size and k-irreducible (for $k \geq 2$).

For $k = 1$ in 1, $\mathcal{C}$ with **free amalgamation** and arity > 1 is 1-overlap closed, recovering Angel, Kechris, and Lyons 2014.

Thoughts on k-overlap closedness II

When the age of $\mathcal{M}$ is not k-overlap closed, it is easy to find non-exchangeable IREs by $\mathcal{C}'$ with growth rate $O(e^{n^{k+\delta}})$:

- linear orders are not 1-overlap closed;
- ages of NIP finitely homogeneous structures are not 1-overlap closed: by Macpherson 1987 they have growth rate $O(e^{n^{1+\delta}})$ and so homogeneous NIP $\mathcal{M}$ has a non-exchangeble IRE by itelf;
- two-graphs are not 2-overlap closed. These are 3-hypergraphs where every four vertices have an even number of hyperedges.

Moral of the story

If $\mathcal{M}$ is k-transitive and "looks random enough" (i.e. has k-overlap closed age), IREs of $\mathcal{M}$ by "essentially k-ary" $\mathcal{C}'$ classes are exchangeable.

Our notion of "random enough" allows for many homogeneous structures with interesting forbidden configurations.
Our notion of "essentially k-ary" is a growth-rate condition on $\mathcal{C}'$.

If $\mathcal{M}$ has some "hidden k-ary structure", then we can find non-exchangeable IREs by $\mathcal{C}'$ of growth rate $O(e^{n^{k+\delta}})$.

Invariant Keisler measures

Definition 9 (Invariant Keisler measure)

An **invariant Keisler measure** (IKM) of $\mathcal{M}$ in the variable x is an $\mathrm{Aut}(M)$-invariant regular Borel probability measure μ on $S_x(M)$.

We are mainly interested in $\mathcal{M}$:

- countable and homogeneous; OR
- ω-saturated and strongly ω-homogeneous (e.g. a monster model).

IKMs give a notion of size on the definable subsets of M.

IKMs in model theory

- Well-behaved in NIP theories (Hrushovski and Pillay 2011; Ensley 1996);
- They naturally arise in many simple theories:
  - pseudofinite fields (Chatzidakis, Van Den Dries, and Macintyre 1992);
  - infinite dimensional vector spaces with forms over finite fields (Cherlin and Hrushovski 2003).

But they are poorly understood in arbitrary simple theories.

IKMs and IREs

IKMs are a special case of IREs.[2]

Example 10 (IKMs of the random graph $\mathcal{R}$)

We can represent any $p \in S_x(\mathcal{R})$ as a colouring of $\mathcal{R}$:

$$B(a) \text{ if and only if } E(x,a) \in p.$$

So, we can view IKMs of $\mathcal{R}$ in the singleton variable as unary IREs. Hence, they are all exchangeable.

More generally, we can choose a canonical language $\mathcal{L}^{\mathrm{pr}}$ such that IKMs of $\mathcal{M}$ in a fixed variable x correspond to IREs of $\mathcal{M}$ to a given space of $\mathcal{L}^{\mathrm{pr}}$-structures.

[2] Adequately defining IREs over arbitrary domains and allowing for the expansion to depend on the underlying structure.

Understanding IKMs of homogeneous structures

- IKMs of homogeneous graphs (and other binary structures) are well-understood since Albert 1994.
    - This is because their IKMs correspond to unary IREs, which are easily understood via De Finetti-like methods, and stability theory (cf. Hrushovski 2012).
- IKMs of homogeneous $(k+1)$-hypergraphs for $k>1$ are harder!
    - this is because graph and hypergraph IREs are harder to understand;
    - but we develop tools precisely for this;
    - e.g. for $\mathcal{M}$ homogeneous k-transitive and $(k+1)$-overlap closed, all IKMs to the variable x are exchangeable.

This has various consequences for understanding IKMs.

Forking and universally measure zero

Recent work on IKMs has focused on comparing two natural model theoretic notions of "smallness" for a definable set:

- $\phi(x, a)$ is **universally measure zero** if it is assigned measure 0 by every (global) IKM. Write $\mathcal{O}(\emptyset)$ for the set of universally measure zero formulas;
- $\phi(x, a)$ **forks over** $\emptyset$. Write $F(\emptyset)$ for the set of formulas forking over $\emptyset$. ▸ Want a reminder of the formal definition of forking?

Forking and universally zero in different contexts

In any theory $F(\emptyset) \subseteq \mathcal{O}(\emptyset)$.

- in **stable** theories $F(\emptyset) = \mathcal{O}(\emptyset)$;
- in **simple** theories, it was recently proved:
  - $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ (Chernikov, Hrushovski, Kruckman, Krupiński, Moconja, Pillay, and Ramsey 2023);
  - $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ in the ω-categorical context (Marimon 2023);
- in **NIP** theories, it was recently proved:
  - $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ (Pillay and Stonestrom 2023);
  - $F(\emptyset) = \mathcal{O}(\emptyset)$ in the ω-categorical context (Braunfeld, Jahel, M. 2024).

Previous examples of $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ are ad-hoc constructions.
Our work shows this phenomenon is pervasive in simple theories satisfying all desirable tameness conditions.

$F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$ in simple homogeneous structures

Corollary 11 (Braunfeld, Jahel and M. 2024)

Let $\mathcal{M}$ be simple, k-transitive, homogeneous in a finite $(k+1)$-ary language, k-overlap closed and with free amalgamation. Then, any IKM of $\mathcal{M}$ in the variable x is exchangeable. Moreover,

1. *EITHER: $\mathrm{Age}(M)$ has n-DAP for all n. In this case there is an IKM assigning positive measure to every non-forking formula;*
2. *OR: $\mathrm{Age}(M)$ fails n-DAP for some n. In this case,*

$$F(\emptyset) \subsetneq \mathcal{O}(\emptyset).$$

For $k > 1$, there are $2^{\aleph_0}$-many structures in 2 (Koponen 2018).

These structures are extremely tame (Conant 2017): supersimple, SU-rank 1, trivial algebraicity, trivial forking, weak elimination of imaginaries.

The generic tetrahedron-free 3-hypergraph

Example 12 (The generic tetrahedron-free 3-hypergraph $\mathcal{H}_4^3$)

$\mathcal{H}_4^3$ is simple, 2-transitive, 2-overlap closed, and with free amalgamation. It fails 4-DAP.

By our results its IKMs are exchangeable. So,

$$\mu(E(x, ab) \wedge E(x, ac) \wedge E(x, bc)) = 0$$

regardless of whether abc forms a hyperedge or not.

But forking is trivial in $\mathcal{H}_4^3$ (Conant 2017),
so if abc does not form a hyperedge $E(x, ab) \wedge E(x, ac) \wedge E(x, bc)$ does not fork over $\emptyset$, giving $F(\emptyset) \subsetneq \mathcal{O}(\emptyset)$.

Problems for the future

Problem 1 (cf. Crane and Towsner 2018)

Can we give representation theorems for the IREs of any arity for more classes of ω-categorical structures?
Some test-cases:

- the generic triangle-free graph (note: our result only says that its unary IREs are exchangeable);
- homogeneous $\mathcal{C}$-relations;
- infinite dimensional vector spaces over finite fields.

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Consistent random expansions

Definition 13 (CRE)

A **consistent random expansion (CRE)** of $\mathcal{C}$ by $\mathcal{C}'$ is an assignment to each $\boldsymbol{H} \in \mathcal{C}$ of a probability distribution $\mathbb{P}_{\boldsymbol{H}}$ on $\mathcal{C}'[H]$ satisfying the following compatibility condition:
let $\phi : \boldsymbol{H} \to \boldsymbol{G}$ be an embedding of structures in $\mathcal{C}$ and $\boldsymbol{H}' \in \mathcal{C}'[H]$. Then,

$$\mathbb{P}_{\boldsymbol{H}}(\boldsymbol{H}') = \mathbb{P}_{\boldsymbol{G}}(\boldsymbol{H}'_{\phi}),$$

where $\boldsymbol{H}'_{\phi}$ is the relabelling of $\boldsymbol{H}'$ according to ϕ.

Back to main presentation

The key lemma for exchangeability

Lemma 14 (Braunfeld, Jahel, and M. 2024)

Suppose that for all $\mathbf{H}_1$, $\mathbf{H}_2 \in \mathcal{C}[k]$, *and* $\epsilon > 0$, *there is some* n, $\mathbf{G} \in \mathcal{C}[n]$ *and non-empty families* Θ_i *of embeddings of* $\mathbf{H}_i$ *in* $\mathbf{G}$ *such that for all* $\mathbf{H}' \in \mathcal{C}'[k]$ *and* $\mathbf{G}' \in \mathcal{C}'[n]$ *we have*

$$\left| \frac{N_{\Theta_1}(\mathbf{H}', \mathbf{G}')}{|\Theta_1|} - \frac{N_{\Theta_2}(\mathbf{H}', \mathbf{G}')}{|\Theta_2|} \right| < \varepsilon,$$

where $N_{\Theta_i}(\mathbf{H}', \mathbf{G}')$ *is the number of embeddings in* Θ_i *that are also embeddings of* $\mathbf{H}'$ *in* $\mathbf{G}'$.
Then every consistent random $\mathcal{C}'$*-expansion* μ *of* $\mathcal{C}$ *is exchangeable.*

▸ Back to main presentation

Forking

Definition 15 (Dividing and forking)

$\phi(x, a)$ **divides over** $\emptyset$ if there is a sequence $(a_i | i < \omega)$ in $\mathrm{tp}(a)$ such that $\{\phi(x, a_i) | i < \omega\}$ is k-inconsistent.

$\phi(x, a)$ **forks over** $\emptyset$ if it implies a disjunction of dividing formulas.

In simple theories forking=dividing. Back to main presentation