Minimal operations over permutation groups

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Outline

Clones

Definition 1 (Clone)

Let B be a (possibly infinite) set.
Let $\mathcal{O}^{(n)} = B^{B^n}$ be the set of functions $f : B^n \to B$, and $\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}$.
We call $\mathcal{C} \subseteq \mathcal{O}$ a (function) **clone** over B if

- $\mathcal{C}$ contains all projections;
- $\mathcal{C}$ is closed under composition.

For $\mathcal{S} \subseteq \mathcal{O}$, $\langle \mathcal{S} \rangle$ is the smallest clone containing $\mathcal{S}$.

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Closed clones

Interested in clones which are **closed** in the **pointwise convergence topology**:[1] For $\mathcal{S} \subseteq \mathcal{O}$,
$f \in \overline{\mathcal{S}} \Leftrightarrow$ for all $A \subseteq B$ finite there is $g \in \mathcal{S}$ such that $g_{\restriction A} = f_{\restriction A}$.

For B finite, topology trivialises (i.e. closed clone=clone).

$\overline{\langle \mathcal{S} \rangle}$ denotes the smallest closed clone containing $\mathcal{S}$.

There is a correspondence between:

- closed clones on B;
- polymorphism clones of relational structures on B.

▸ Definition of polymorphism clone

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Monoidal intervals

Let $\mathcal{T}$ be a **transformation monoid** on B
(i.e. unary operations containing Id, and closed under composition).

Closed clones whose unary operations are $\overline{\mathcal{T}}$ form an interval in the lattice of closed clones on B, known as the **monoidal interval of $\mathcal{T}$**.

Studying the structure and size of monoidal intervals has a long history in universal algebra:

- for $\mathcal{O}_B^{(1)}$ (Burle 1967);
- for $G \curvearrowright B$ a permutation group (Pálfy and Szendrei 1982; Kearnes and Szendrei 2001) (with focus on collapse);
- for other monoids (Krokhin 1995);
- over infinite sets (Pinsker 2008).

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Minimal operations

What is the minimal amount of structure in a clone containing $\mathcal{T}$?

Definition 2 (Minimal clone)

Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed clones.
$\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.

Definition 3 (almost minimal and minimal operations)

The k-ary operation $f \in \mathcal{O} \setminus \mathcal{C}$ is **almost minimal above** $\mathcal{C}$ if for each $r < k$,

$$\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}.$$

If f is almost minimal above $\mathcal{C}$ and $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$, then f is **minimal above** $\mathcal{C}$.

Minimal operations

What is the minimal amount of structure in a clone containing $\mathcal{T}$?

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Basic facts on minimality and almost minimality

- $\mathcal{D}$ is minimal above $\mathcal{C}$ if and only if $\mathcal{D} = \overline{\langle \mathcal{C} \cup \{f\} \rangle}$ for f minimal above $\mathcal{C}$;
- minimal elements in the interval of $\mathcal{T}$ above $\overline{\langle \mathcal{T} \rangle}$ correspond to minimal clones above $\overline{\langle \mathcal{T} \rangle}$ which are not essentially unary[2];
- ALWAYS, if $\mathcal{E} \supsetneq \mathcal{C}$, there is $f \in \mathcal{E} \setminus \mathcal{C}$ almost minimal above $\mathcal{C}$;

We will study minimal operations above $\overline{\langle G \rangle}$ for $G \curvearrowright B$ a non-trivial permutation group.

[2] f is **essentially unary** if it depends on only one variable. Otherwise, it is **essential**.

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Oligomorphic permutation groups

We are particularly interested in oligomorphic permutation groups:

Definition 4 (Oligomorphicity, ω-categoricity)

B countably infinite. $G \curvearrowright B$ is **oligomorphic** if $G \curvearrowright B^n$ has finitely many orbits for each $n \in \mathbb{N}$;
A first-order structure $\mathbb{B}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{B}) \curvearrowright B$ is oligomorphic.

Examples of ω-categorical structures:

- $(\mathbb{N}, =)$;
- $(\mathbb{Q}, <)$;
- The Random graph;
- Countable vector spaces over finite fields.

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Fact 5 (Existence of minimal operations)

Let $\mathcal{C} \subsetneq \mathcal{D}$ be closed function clones. Suppose either

- *B is finite; or*
- *$\mathcal{C} = \overline{\langle \mathrm{Aut}(\mathbb{B}) \rangle}$ for $\mathbb{B}$ ω-categorical in a finite relational language.*

Then, there is $\mathcal{E} \subseteq \mathcal{D}$ which is minimal above $\mathcal{C}$.

Note: in general, this can fail when B is infinite.

Rosenberg's five types theorem

Theorem 6 (Five types theorem, Rosenberg 1986)

Let B be finite and f be minimal above $\langle \mathrm{Id} \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary majority operation;*
4. *a minority of the form $x + y + z$ in some Boolean group $(B, +)$;*
5. *a k-ary semiprojection for some $k \geq 3$.*

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Ternary majority: an operation $m : B^3 \to B$ such that

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx m(x, x, x) \approx x;$$

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A group is **Boolean** if every non-identity element has order 2.
Ternary minority: an operation $\mathfrak{m} : B^3 \to B$ such that

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Semiprojection: $f : B^k \to B$ such that there is an $i \in \{1, \ldots, k\}$ such that whenever $(a_1, \ldots, a_k)$ is a non-injective tuple from B,

$$f(a_1, \ldots, a_k) = a_i.$$

The oligomorphic case

Bodirsky and Chen 2007 classify minimal operations above oligomorphic permutation groups.

Theorem 7 (Oligomorphic case, Bodirsky and Chen 2007)

Let $G \curvearrowright B$ be an oligomorphic permutation group. Let f be minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-majority operation;*
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Ternary quasi-majority:[3] an operation $m : \mathbb{B}^3 \to \mathbb{B}$ such that

$$m(x, x, y) \approx m(x, y, x) \approx m(y, x, x) \approx m(x, x, x) \not\approx x;$$

[3] "quasi":= we don't ask that $f(x, \ldots, x) \approx x$.

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Quasi-semiprojection: $f : B^k \to B$ such that there is an $i \in \{1, \ldots, k\}$ and $g \in \overline{G}$ such that whenever $(a_1, \ldots, a_k)$ is a non-injective tuple,

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Note: No minority type!

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We obtain better results even in this case!

The Main Theorem

Theorem 8 (Minimal operation, MP 2024)

Let $G \curvearrowright B$ be non-trivial with s many orbits (s possibly infinite). Let f be a minimal operation above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-minority operation of the form $\alpha\mathfrak{q}$ for $\alpha \in G$, where*
   - *G is a Boolean group acting freely on B with $s = 2^n$ or infinite;*
   - *the operation $\mathfrak{q}$ is a G-invariant Boolean Steiner 3-quasigroup.*
4. *f is a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

- type 3 strengthens quasi-minority. Essentially never occurs;
- type 4 strengthens quasi-semiprojection. Arity bounded by orbits;
- No quasi-majorities!

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In the oligomorphic case we improve on Bodirsky and Chen 2007:

- We reduced from four to three types ($G \curvearrowright B$ is not free);
- Stronger characterisation of the quasi-semiprojection case.

G-invariant Boolean Steiner 3-quasigroups

Definition 9

A G-invariant Boolean **Steiner 3-quasigroup**[3] is a symmetric ternary minority operation $\mathfrak{q}$ satisfying:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \; ; \qquad \text{(SQS)}$$

$$\mathfrak{q}(x, y, \mathfrak{q}(z, y, w)) \approx \mathfrak{q}(x, z, w) \; ; \qquad \text{(Bool)}$$

$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z). \qquad \text{(Inv)}$$

[3] Studied in universal algebra (Quackenbush 1975; Ganter and Werner 1975) and design theory (Lindner and Rosa 1978).

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$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z). \qquad \text{(Inv)}$$

(SQS) yields that $\mathfrak{q}(x_1, x_2, x_3) = x_4 \wedge \bigwedge_{i<j\leq 4} x_i \neq x_j$ is a **Steiner quadruple system** on B: a 4-hypergraph on B such that every three vertices are in a unique 4-hyperedge.

Steiner 3-quasigroups correspond to Steiner quadruple systems on B.

[3] Studied in universal algebra (Quackenbush 1975; Ganter and Werner 1975) and design theory (Lindner and Rosa 1978).

G-invariant Boolean Steiner 3-quasigroups

Definition 9

A G-invariant **Boolean Steiner 3-quasigroup**[3] is a symmetric ternary minority operation $\mathfrak{q}$ satisfying:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \; ; \qquad \text{(SQS)}$$

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Correspond to $x + y + z$ on a Boolean group on B.

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$\mathfrak{q}(x, y, z)$ also induces a Boolean Steiner 3-quasigroup on the G-orbits.

Description of minimal minorities

Theorem 10 (Description of minimal minorities, MP 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s many orbits.

- *All G-invariant Boolean Steiner 3-quasigroups are minimal above $\overline{\langle G \rangle}$;*
- *There are G-invariant Boolean Steiner 3-quasigroups if and only if $s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
- *For $s = 2^n$, the number of G-invariant Boolean Steiner 3-quasigroups is 1 for $n = 0$, and for $n \geq 1$ it is*

$$\frac{(2^n - 1)!|G|^{(2^n - n - 1)}}{\prod_{k=0}^{n-1}(2^n - 2^k)} .$$

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Counting G-invariant Boolean Steiner 3-quasigroups requires solving a small problem in projective geometry.

Orbit-semiprojections

f is an **orbit-semiprojection** if there is an $i \in \{1, \ldots, k\}$ and $g \in \overline{G}$ such that whenever at least two of the a_j lie in the same orbit,

$$f(a_1, \ldots, a_k) = g(a_i).$$

- Almost minimal k-ary orbit-semiprojections exist for all $2 \leq k \leq s$;
- Pálfy 1986: k-ary semiprojections minimal above $\langle \mathrm{Id} \rangle$ exist for each $2 \leq k \leq |B|$.

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Theorem 11 (Pálfy's theorem for orbit-semiprojections, MP 2024)

Let $G \curvearrowright B$ with s-many orbits be finite or oligomorphic. Then, for all $2 \leq k \leq s$, there is a k-ary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.

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When G is finite (non-trivial), taking $b \in B$ in an orbit of size > 1, let

$$f(a_1, \ldots, a_k) := \begin{cases} b \text{ if } a_1 \sim b \text{ and } a_1, \ldots, a_k \text{ are all in distinct orbits;} \\ a_1 \text{ otherwise.} \end{cases}$$

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When $G \curvearrowright B$ is oligomorphic, taking $b \in B$ in an orbit of size > 1, letting γ belong to a minimal closed $(\overline{G}_b, \overline{G})$-biact,[3]

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[3] A $(\overline{G}_b, \overline{G})$-biact is a set $\mathcal{I} \subseteq \overline{G}$ such that $\overline{G}_b \mathcal{I} \overline{G} \subseteq \mathcal{I}$

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- always holds for minimality in the lattice of *all* clones (rather than closed clones);
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Methods: almost minimality

To classify minimal operations, we first classify almost minimal operations above $\overline{\langle G \rangle}$:

- Start from a weak version of Rosenberg's Theorem for almost minimal operations above a monoid without constant operations (implicit in Bodirsky and Chen 2007);
- Show that certain behaviours cannot be witnessed by almost minimal operations since they give rise to non-injective unary operations.

This splits into three cases:

- $G \curvearrowright B$ not a Boolean group acting freely on B;
- $G \curvearrowright B$ a Boolean group acting freely on B with $|G| > 2$;
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Case1: The three types

Lemma 12 (Three types lemma, MP 2024)

Let $G \curvearrowright B$ with s-many orbits be such that G is not a Boolean group acting freely on B. Let f be almost minimal above $\overline{\langle G \rangle}$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

Case 2: the Boolean case

Lemma 13 (Boolean case, MP 2024)

Let $G \curvearrowright B$ be a Boolean group acting freely on B with s-many orbits and $|G| > 2$. Let f be an almost minimal operation above $\langle G \rangle$. Then, f is one of:

1. *a unary operation;*
2. *a binary operation;*
3. *a G-quasi-minority;*
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

A ***G*-quasi-minority** is a ternary operation such that for all $\beta \in G$,

$$\mathfrak{m}(y, x, \beta x) \approx \mathfrak{m}(x, \beta x, y) \approx \mathfrak{m}(x, y, \beta x) \approx \mathfrak{m}(\beta y, \beta y, \beta y).$$

Case 3: the $\mathbb{Z}_2$ case

Lemma 14 ($\mathbb{Z}_2$ case, MP 2024)

Let $\mathbb{Z}_2$ act freely on B with s-many orbits. Let f be an almost minimal operation above $\langle \mathbb{Z}_2 \rangle$. Then, f is one of

1. *a unary operation;*
2. *a G-quasi-minority;*
3. *an odd majority;*
4. *an odd Malcev, up to permuting variables;*
5. *a k-ary orbit-semiprojection for $2 \leq k \leq s$.*

An **odd majority** m is a ternary quasi-majority such that for $\gamma \neq \mathrm{Id}$ in $\mathbb{Z}_2$,

$$m(y, x, \gamma x) \approx m(x, \gamma x, y) \approx m(x, y, \gamma x) \approx m(y, y, y).$$

An **odd Malcev** is such that $M(x, \gamma y, z)$ is an odd majority.
Odd majorities and odd Malcev cannot be minimal!

Motivation from CSPs

$\tau :=$ finite relational language.
$\mathbb{B} :=$ a fixed τ-structure.

Definition 15 ($\mathrm{CSP}(\mathbb{B})$)

$\mathrm{CSP}(\mathbb{B})$ is the following computational problem:

- **INPUT**: A **finite** τ-structure $\mathbb{A}$;
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We study $\mathrm{CSP}(\mathbb{B})$ for $\mathbb{B}$ finite or ω-categorical.

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The algebraic approach to CSPs

Algebraic approach to CSPs:

The polymorphisms of $\mathbb{B}$ capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Highly successful in the finite setting:

Theorem 16 (Bulatov 2017; Zhuk 2017)

Let $\mathbb{B}$ be finite. Then:

- *EITHER $\mathbb{B}$ has a Siggers polymorphism.*[3]
  In this case, $\mathrm{CSP}(\mathbb{B})$ is in P;
- *OR $\mathbb{B}$ "pp-constructs" EVERYTHING (i.e., all finite structures)*
  In this case, $\mathrm{CSP}(\mathbb{B})$ is NP-complete.

[3] A polymorphism $s : \mathbb{B}^6 \to \mathbb{B}$ such that

$$s(x, y, x, z, y, z) \approx s(y, x, z, x, z, y).$$

The algebraic approach to CSPs

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The polymorphisms of $\mathbb{B}$ capture the computational complexity of $\mathrm{CSP}(\mathbb{B})$.

Often successful for $\mathbb{B}$ ω-categorical: complexity dichotomies for CSPs of structures first-order definable in:

- $(\mathbb{Q}, <)$ (Bodirsky and Kára 2010);
- homogeneous graphs (Bodirsky, Martin, Pinsker, and Pongrácz 2019);
- countable unary structures (Bodirsky and Mottet 2018);

$\vdots$

Bodirsky-Pinsker conjecture: CSPs of a large class of ω-categorical structures[3] satisfy a complexity dichotomy analogous to the finite-domain one.

[3] First-order reducts of finitely bounded homogeneous structures.

Understanding low arity polymorphisms

Question 1

What is the minimal amount of structure in $\mathrm{Pol}(\mathbb{B})$ when $\mathrm{CSP}(\mathbb{B})$ is not NP-hard (due to pp-constructing EVERYTHING)?

- Sufficient to consider case of a (model complete) **core**: $\overline{\mathrm{Aut}(\mathbb{B})} = \mathrm{End}(\mathbb{B})$;
- We can assume $\mathrm{Pol}(\mathbb{B})$ is **essential**: if $\mathrm{Pol}(\mathbb{B})$ is NOT essential, then $\mathbb{B}$ pp-interprets EVERYTHING;
- **Bottom-up approach to CSPs:** several complexity classifications identify the behaviours of low arity essential polymorphisms.

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Binary essential polymorphisms

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(Bodirsky and Kára 2008):

Lemma 5. *Every essentially at least binary operation together with all permutations locally generates a binary operation that depends on both arguments.*

(Bodirsky and Kára 2010)

Lemma 10. *Let Γ be a relational structure and let R be a k-ary relation that is a union of l orbits of k-tuples of $Aut(\Gamma)$. If R is violated by a polymorphism g of Γ of arity $m \geq l$, then R is also violated by an l-ary polymorphism of Γ.*

(Bodirsky and Pinsker 2014):

LEMMA 40: *Let $f\colon V^k \to V$ be an essential operation. Then f generates a binary essential operation.*

(Bodirsky 2021; Mottet and Pinsker 2024):

LEMMA 6.1.29. *Let $\mathscr{C}$ be a clone with an essential operation that contains a permutation group $\mathscr{G}$ with the orbital extension property. Then $\mathscr{C}$ must also contain a binary essential operation.*

PROPOSITION 23. *Let $\mathbb{A}$ be a first-order reduct of a homogeneous structure $\mathbb{B}$ such that $\mathbb{B}$ has a free orbit. If $\mathrm{Pol}(\mathbb{A})$ contains an essential function, then it contains a binary essential operation.*

(Mottet, Nagy, and Pinsker 2024)

Lemma 27. *Let $\mathbb{A}$ be a first-order reduct of $\mathbb{H}$ that is a model-complete core. If $\mathrm{Pol}(\mathbb{A})$ does not have a uniformly continuous clone homomorphism to $\mathscr{P}$, then it contains a binary essential operation.*

Also done in:

- Bodirsky, Jonsson, and Van Pham 2017;
- Bodirsky and Mottet 2018;
- Kompatscher and Van Pham 2018;
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ISSUE: These techniques are ad-hoc!

A question of Bodirsky

Question 2 (Question 24 in Bodirsky 2021)

Suppose $\mathbb{B}$ is an ω-categorical (model complete) core and $\mathrm{Pol}(\mathbb{B})$ has an essential polymorphism.
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Answer: No.(Though true if $\mathrm{Aut}(\mathbb{B})$ has ≤ 2 orbits.)
Counterexample: for any oligomorphic $\mathrm{Aut}(\mathbb{B}) \curvearrowright \mathbb{B}$ with s orbits for $s \geq 3$, there is an ω-categorical structure $\mathbb{B}'$ such that

$$\mathrm{Pol}(\mathbb{B}') := \overline{\langle \mathrm{Aut}(\mathbb{B}) \cup \{f\} \rangle},$$

where f is an essential s-ary orbit-semiprojection.

Why easy problems lie above binary operations

Moral answer (for the purposes of CSPs): Yes!

Theorem 16 (Finding binary operations, MP 2024)

Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Suppose that $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G}$, and

($\star$) *$\mathcal{C}$ has no uniformly continuous clone homomorphism[4] to $\mathcal{P}_{\{0,1\}}$, the clone of projections on a two-element set.*

Then, $\mathcal{C}$ contains a binary essential operation.

[4] $\xi : \mathcal{C} \to \mathcal{P}_{\{0,1\}}$ is **uniformly continuous** if there is some finite $B' \subseteq B$ such that $f_{\restriction B'} = g_{\restriction B'}$ implies $\xi(f) = \xi(g)$.

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For $\mathcal{C} = \mathrm{Pol}(\mathbb{B})$ and $\mathbb{B}$ finite or ω-categorical:

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- $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G} \Leftrightarrow \mathbb{B}$ is a (model complete) core;
- ($\star$)$\Leftrightarrow \mathbb{B}$ does NOT pp-interpret EVERYTHING.

[4] $\xi : \mathcal{C} \to \mathcal{P}_{\{0,1\}}$ is **uniformly continuous** if there is some finite $B' \subseteq B$ such that $f_{\restriction B'} = g_{\restriction B'}$ implies $\xi(f) = \xi(g)$.

Why easy problems lie above binary operations

Theorem 16 (Finding binary operations, MP 2024)

Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Suppose that $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G}$, and

($\star$) *$\mathcal{C}$ has no uniformly continuous clone homomorphism to $\mathcal{P}_{\{0,1\}}$, the clone of projections on a two-element set.*

Then, $\mathcal{C}$ contains a binary essential operation.

Proof idea.

Suppose by contrapositive that $\mathcal{C} \cap \mathcal{O}^{(2)} = \overline{\langle G \rangle} \cap \mathcal{O}^{(2)}$.

Then, all ternary operations in $\mathcal{C}$ are almost minimal.

So $\mathcal{C} \cap \mathcal{O}^{(3)}$=essentially unary operations and orbit-semiprojections.

These will only satisfy trivial identities, which is sufficient to build a uniformly continuous homomorphism to $\mathcal{P}_{\{0,1\}}$. □

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Thank you!

A brief recap:

- We classify minimal (and almost minimal) operations above arbitrary permutation groups;
- We get fewer types than those of Rosenberg's theorem above the trivial group;
- We give general reasons why when $\mathbb{B}$ is an ω-categorical (model complete) core and $\mathrm{CSP}(\mathbb{B})$ is not NP-hard, we can find binary essential polymorphisms;

QR code to preprint:

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Polymorphisms

Let $\mathbb{B}$ be a relational structure.

Definition 17 (Polymorphisms)

$f : \mathbb{B}^n \to \mathbb{B}$ is a **polymorphism** if it **preserves all relations** of $\mathbb{B}$:

$$\begin{pmatrix} a_1^1 \\ \vdots \\ a_k^1 \end{pmatrix}, \dots, \begin{pmatrix} a_1^n \\ \vdots \\ a_k^n \end{pmatrix} \in R^{\mathbb{B}} \Rightarrow \begin{pmatrix} f(a_1^1, \dots, a_1^n) \\ \vdots \\ f(a_k^1, \dots, a_k^n) \end{pmatrix} \in R^{\mathbb{B}}.$$

We call $\mathbf{Pol}(\mathbb{B})$ the set of polymorphisms of $\mathbb{B}$.
The **polymorphism clone of** $\mathbb{B}$.

- Unary polymorphism = endomorphism;
- **Projections** to one coordinate **are always polymorphisms**.

Back to main presentation

pp-interpretations and pp-constructions

A ***pp*-formula** is a first-order formula consisting only of existential quantifiers, conjunctions, and atomic formulas.

Definition 18 (pp-interpretation, pp-construction)

$\mathbb{B}$ ***pp*-interprets** $\mathbb{A}$ if there is partial surjective $h : \mathbb{B}^d \to \mathbb{A}$ such that for every $R \subseteq \mathbb{A}^n$ that is a relation of $\mathbb{A}$ (or $\mathbb{A}$, or equality on $\mathbb{A}$), $h^{-1}(R)$ is defined by a pp-formula in $\mathbb{B}^{nd}$.

$\mathbb{B}$ ***pp*-constructs** $\mathbb{A}$ if it is homomorphically equivalent to a structure that pp-interprets $\mathbb{A}$.