1 Preliminaries

1.1 Clones and minimal operations

Let B be a (possibly infinite) set.
For $n \in \mathbb{N}$, $\mathcal{O}^{(n)}$ is the set B^{B^n} of functions $B^n \to B$.
$\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}$.

Definition 1 (Function clone)**.** A **function clone** on B is a set $\mathcal{C} \subseteq \mathcal{O}$ containing all projections and closed under composition of functions.

Definition 2 (Notions of closure)**.** For $\mathcal{S} \subseteq \mathcal{O}$,
$\langle \mathcal{S} \rangle$ is the smallest function clone containing $\mathcal{S}$.
We study **closed clones** with respect to the **pointwise convergence topology**: for $\mathcal{S} \subseteq \mathcal{O}$, $f \in \overline{\mathcal{S}}$ if for each finite $A \subseteq B$, there is some $g \in \mathcal{S}$ such that $g|_A = f|_A$.
$\overline{\langle \mathcal{S} \rangle}$ is the smallest closed function clone containing $\mathcal{S}$.

We are interested in closed function clones because they correspond to **polymorphism clones** of relational structures.
Still, our results also hold in the lattice of (not necessarily closed) function clones.

Definition 3. Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed function clones.
We say that $\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed function clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.
The k-ary operation $f \in \mathcal{O} \setminus \mathcal{C}$ is **almost minimal** above $\mathcal{C}$ if, for each $r < k$, $\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}$.
We say that $f \in \mathcal{O} \setminus \mathcal{C}$ is **minimal above** $\mathcal{C}$ if it is almost minimal above $\mathcal{C}$ and $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$.

Fact 4. *A closed function clone $\mathcal{D}$ is minimal above $\mathcal{C}$ if and only if there is some operation f which is minimal above $\mathcal{C}$ and such that $\overline{\langle \mathcal{C} \cup \{f\} \rangle} = \mathcal{D}$.*

Definition 5 (oligomorphicity, ω-categoricity)**.** B countably infinite. $G \curvearrowright B$ is **oligomorphic** if $G \curvearrowright B^n$ has finitely many orbits for each $n \in \mathbb{N}$.
A first-order structure $\mathbb{B}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{B}) \curvearrowright B$ is oligomorphic.
Examples: $(\mathbb{N}, =)$, $(\mathbb{Q}, <)$, countable vector spaces over finite fields.

Fact 6. *Let $\mathcal{C} \subsetneq \mathcal{D}$ be closed function clones.*
Suppose either

- *B is finite; or*
- *$\mathcal{C} = \overline{\langle \mathrm{Aut}(\mathbb{B}) \rangle}$ for $\mathbb{B}$ an ω-categorical structure in a finite relational language.*

Then, there is $\mathcal{E} \subseteq \mathcal{D}$ which is minimal above $\mathcal{C}$.

1.2 Rosenberg's Five Types Theorem and friends

Definition 7.

- a **ternary quasi-majority** is a ternary operation m such that
$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x)\ ;$$
- a **ternary quasi-minority** is a ternary operation $\mathfrak{m}$ such that
$$\mathfrak{m}(x,x,y) \approx \mathfrak{m}(x,y,x) \approx \mathfrak{m}(y,x,x) \approx \mathfrak{m}(y,y,y)\ ;$$
- a **quasi-semiprojection** is a k-ary operation f such that there is an $i \in \{1,\ldots,k\}$ and a unary operation g such that whenever $(a_1,\ldots,a_k)$ is a non-injective tuple from B,
$$f(a_1,\ldots,a_k) = g(a_i)\ .$$

We remove the prefix "quasi" when the operation is **idempotent**, i.e., satisfies $f(x,\ldots,x) \approx x$; in the case of a semiprojection, idempotency implies $g(x) \approx x$.

Theorem 8 (Five Types Theorem [3])**.** *Let B be finite and let f be a minimal operation above $\langle \mathrm{Id} \rangle$. Then f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary majority operation;*
4. *a ternary minority operation of the form $x+y+z$ in a Boolean group $(B,+)$;*
5. *a k-ary semiprojection for some $k \geq 3$.*

A **Boolean group** (a.k.a. elementary Abelian 2-group) is a group where every non-identity element has order 2. They are just direct sums of copies of $\mathbb{Z}_2$.

Theorem 9 (Four types, oligomorphic case [1])**.** *Let $G \curvearrowright B$ be an oligomorphic permutation group. Let f be minimal above $\overline{\langle G \rangle}$. Then, f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-majority operation;*
4. *a k-ary quasi-semiprojection for some $3 \leq k \leq 2r - s$, where r is the number of G-orbitals (orbits under the componentwise action of G on pairs) and s is the number of G-orbits.*

2 Minimal operations over permutation groups

We classify minimal operations above $\overline{\langle G \rangle}$ for $G \curvearrowright B$ an arbitrary non-trivial permutation group.

Our proof strategy: first classify almost minimal operations above $\overline{\langle G \rangle}$.

We use a weak version of Theorem 8 for almost minimal operations above a monoid (implicit in [1]) and show that certain behaviours cannot be witnessed by almost minimal operations since they give rise to non-injective unary operations.

Main Theorem 10 (Minimal operations over permutation groups [2]). *Let $G \curvearrowright B$ be a non-trivial permutation group with s many orbits (where s is possibly infinite). Let f be a minimal operation above $\overline{\langle G \rangle}$. Then, f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-minority operation of the form $\alpha\mathfrak{q}$ for $\alpha \in G$, where*
   - *G is a Boolean group acting freely on B;*
   - *$s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
   - *the operation $\mathfrak{q}$ is a G-invariant Boolean Steiner 3-quasigroup.*
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

Remark 11. Quasi-majorities do not appear!

Minimal quasi-minorites almost never occur and are completely understood;

We specify the behaviour of quasi-semiprojections on the orbits and bound their arity by the number of orbits rather than the number of orbitals.

Definition 12. $G \curvearrowright B$ is **free** if the only group element fixing any element of B is the identity (i.e., $ga = a$ implies $g = 1$).

Definition 13. Let $G \curvearrowright B$. The k-ary operation f on B is an **orbit-semiprojection** if there is $i \in \{1, \ldots, k\}$ and a unary operation $g \in \overline{G}$ such that for any tuple $(a_1, \ldots, a_k)$ where at least two of the a_j lie in the same G-orbit,

$$f(a_1, \ldots, a_k) = g(a_i) \; .$$

Definition 14. A **G-invariant Boolean Steiner 3-quasigroup** is a symmetric ternary minority operation $\mathfrak{q}$ also satisfying the following conditions:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \; ; \qquad \text{(SQS)}$$

$$\mathfrak{q}(x, y, \mathfrak{q}(z, y, w)) \approx \mathfrak{q}(x, z, w) \; ; \qquad \text{(Bool)}$$

$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z) \; . \qquad \text{(Inv)}$$

Theorem 15 (Description of minimal minorities [2]). *Let $G \curvearrowright B$ be a Boolean group acting freely on B with s many orbits.*

- *All G-invariant Boolean Steiner 3-quasigroups are minimal above $\overline{\langle G \rangle}$;*
- *There are G-invariant Boolean Steiner 3-quasigroups if and only if $s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
- *For $s = 2^n$, the number of G-invariant Boolean Steiner 3-quasigroups is 1 for $n = 0$, and for $n \geq 1$ it is*

$$\frac{(2^n - 1)!|G|^{(2^n - n - 1)}}{\prod_{k=0}^{n-1}(2^n - 2^k)} \; .$$

Theorem 16 (Pálfy's Theorem for orbit-semiprojections [2]). *Let $G \curvearrowright B$ with s-many orbits be finite or oligomorphic. Then, for all $2 \leq k \leq s$, there is a k-ary orbit-semiprojection minimal above $\overline{\langle G \rangle}$.*

2.1 Finding binary essential operations

Definition 17. We say that a k-ary operation f is **essentially unary** if it depends on at most one variable. Otherwise, we say that f is **essential**.

Definition 18. A map $\eta : \mathcal{C} \to \mathcal{D}$ is a **clone homomorphism** if it preserves arities and universally quantified identities.

For D finite, η is **uniformly continuous** if there exists a finite $A \subseteq C$ such that $f_{\restriction A} = g_{\restriction A}$ implies $\eta(f) = \eta(g)$ for all $f, g \in \mathcal{C}$.

Theorem 19 (Finding binary operations, [2]). *Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Suppose that $\mathcal{C} \cap \mathcal{O}^{(1)} = \overline{G}$, and that $\mathcal{C}$ has no uniformly continuous clone homomorphism to $\mathcal{P}_{\{0,1\}}$, the clone of projections on a two-element set. Then, $\mathcal{C}$ contains a binary essential operation.*

Theorem 19 has applications to the study of CSPs of ω-categorical structures:

if $\mathbb{B}$ is a model complete core which does not pp-interpret EVERYTHING, then $\mathrm{Pol}(\mathbb{B})$ has a binary essential operation.

References

[1] M. Bodirsky, H. Chen. Oligomorphic clones. Algebra Universalis 57 (2007).

[2] P. Marimon, M. Pinsker. Minimal operations over permutation groups. 2024. arXiv: 2410.22060 [math.RA].

[3] I. G. Rosenberg. "Minimal clones I: the five types". Lectures in universal algebra. Elsevier, 1986.