1 Preliminaries

1.1 Clones and minimal operations

Let B be a (possibly infinite) set.
For $n \in \mathbb{N}$, $\mathcal{O}^{(n)}$ is the set B^{B^n} of functions $B^n \to B$.
$\mathcal{O} := \bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}$.

Definition 1 (Function clone)**.** A **function clone** on B is a set $\mathcal{C} \subseteq \mathcal{O}$ containing all projections and closed under composition of functions.

Definition 2 (Notions of closure)**.** For $\mathcal{S} \subseteq \mathcal{O}$,
$\langle \mathcal{S} \rangle$ is the smallest function clone containing $\mathcal{S}$.
We study clones which are **closed** with respect to the **pointwise convergence topology**: for $\mathcal{S} \subseteq \mathcal{O}$, $f \in \overline{\mathcal{S}}$ if for each finite $A \subseteq B$, there is some $g \in \mathcal{S}$ such that $g|_A = f|_A$.
$\overline{\langle \mathcal{S} \rangle}$ is the smallest closed function clone containing $\mathcal{S}$.

We are interested in closed function clones because they correspond to **polymorphism clones** of relational structures.
Still, our results also hold in the lattice of (not necessarily closed) function clones.

Definition 3. Let $\mathcal{D} \supsetneq \mathcal{C}$ be closed function clones.
We say that $\mathcal{D}$ is **minimal above** $\mathcal{C}$ if there is no closed function clone $\mathcal{E}$ such that $\mathcal{C} \subsetneq \mathcal{E} \subsetneq \mathcal{D}$.
The k-ary operation $f \in \mathcal{O} \setminus \mathcal{C}$ is **almost minimal** above $\mathcal{C}$ if, for each $r < k$, $\overline{\langle \mathcal{C} \cup \{f\} \rangle} \cap \mathcal{O}^{(r)} = \mathcal{C} \cap \mathcal{O}^{(r)}$.
We say that $f \in \mathcal{O} \setminus \mathcal{C}$ is **minimal above** $\mathcal{C}$ if it is almost minimal above $\mathcal{C}$ and $\overline{\langle \mathcal{C} \cup \{f\} \rangle}$ is minimal above $\mathcal{C}$.

Fact 4. *A closed function clone $\mathcal{D}$ is minimal above $\mathcal{C}$ if and only if there is some operation f which is minimal above $\mathcal{C}$ and such that $\overline{\langle \mathcal{C} \cup \{f\} \rangle} = \mathcal{D}$.*

Definition 5 (oligomorphicity, ω-categoricity)**.** B countably infinite. $G \curvearrowright B$ is **oligomorphic** if $G \curvearrowright B^n$ has finitely many orbits for each $n \in \mathbb{N}$.
A first-order structure $\mathbb{B}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{B}) \curvearrowright B$ is oligomorphic.
Examples: $(\mathbb{N}, =)$, $(\mathbb{Q}, <)$, countable vector spaces over finite fields.

Fact 6. *Let $\mathcal{C} \subsetneq \mathcal{D}$ be closed function clones.*
Suppose either

- *B is finite; or*
- *$\mathcal{C} = \overline{\langle \mathrm{Aut}(\mathbb{B}) \rangle}$ for $\mathbb{B}$ an ω-categorical structure in a finite relational language.*

Then, there is $\mathcal{E} \subseteq \mathcal{D}$ which is minimal above $\mathcal{C}$.

1.2 Rosenberg's Five Types Theorem and friends

Definition 7.

- a **ternary quasi-majority** is a ternary operation m such that
$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx m(x,x,x) \ ;$$
- a **ternary quasi-minority** is a ternary operation $\mathfrak{m}$ such that
$$\mathfrak{m}(x,x,y) \approx \mathfrak{m}(x,y,x) \approx \mathfrak{m}(y,x,x) \approx \mathfrak{m}(y,y,y) \ ;$$
- a **quasi-semiprojection** is a k-ary operation f such that there is an $i \in \{1,\dots,k\}$ and a unary operation g such that whenever $(a_1,\dots,a_k)$ is a non-injective tuple from B,
$$f(a_1,\dots,a_k) = g(a_i) \ .$$

We remove the prefix "quasi" when the operation is **idempotent**, i.e., satisfies $f(x,\dots,x) \approx x$; in the case of a semiprojection, idempotency implies $g(x) \approx x$.

Theorem 8 (Five Types Theorem [3])**.** *Let B be finite and let f be a minimal operation above $\langle \mathrm{Id} \rangle$. Then f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary majority operation;*
4. *a ternary minority operation of the form $x+y+z$ in a Boolean group $(B,+)$;*
5. *a k-ary semiprojection for some $k \geq 3$.*

A **Boolean group** (a.k.a. elementary Abelian 2-group) is a group where every non-identity element has order 2. They are just direct sums of copies of $\mathbb{Z}_2$.

Theorem 9 (Four types, oligomorphic case [1])**.** *Let $G \curvearrowright B$ be an oligomorphic permutation group. Let f be minimal above $\overline{\langle G \rangle}$. Then, f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-majority operation;*
4. *a k-ary quasi-semiprojection for some $3 \leq k \leq 2r-s$, where r is the number of G-orbitals (orbits under the componentwise action of G on pairs) and s is the number of G-orbits.*

2 Minimal operations over permutation groups

We classify minimal operations above $\overline{\langle G\rangle}$ for $G \curvearrowright B$ an arbitrary non-trivial permutation group.

Main Theorem 10 (Minimal operations over permutation groups [2]). *Let $G \curvearrowright B$ be a non-trivial permutation group with s many orbits (where s is possibly infinite). Let f be a minimal operation above $\overline{\langle G\rangle}$. Then, f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a ternary quasi-minority operation of the form $\alpha\mathfrak{q}$ for $\alpha \in G$, where*
   - *G is a Boolean group acting freely on B;*
   - *$s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
   - *the operation $\mathfrak{q}$ is a G-invariant Boolean Steiner 3-quasigroup.*
4. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

Remark 11. Quasi-majorities do not appear!
Minimal quasi-minorites almost never occur and are completely understood (Theorem 15);
We specify the behaviour of quasi-semiprojections on the orbits and bound their arity by the number of orbits rather than the number of orbitals.

Definition 12. $G \curvearrowright B$ is **free** if the only group element fixing any element of B is the identity (i.e., $ga = a$ implies $g = 1$).

Definition 13. Let $G \curvearrowright B$. The k-ary operation f on B is an **orbit-semiprojection** if there is $i \in \{1, \ldots, k\}$ and a unary operation $g \in \overline{G}$ such that for any tuple $(a_1, \ldots, a_k)$ where at least two of the a_j lie in the same G-orbit,

$$f(a_1, \ldots, a_k) = g(a_i) \ .$$

Definition 14. A ***G*-invariant Boolean Steiner 3-quasigroup** is a symmetric ternary minority operation $\mathfrak{q}$ also satisfying the following conditions:

$$\mathfrak{q}(x, y, \mathfrak{q}(x, y, z)) \approx z \ ; \qquad \text{(SQS)}$$
$$\mathfrak{q}(x, y, \mathfrak{q}(z, y, w)) \approx \mathfrak{q}(x, z, w) \ ; \qquad \text{(Bool)}$$
$$\text{for all } \alpha, \beta, \gamma \in G, \mathfrak{q}(\alpha x, \beta y, \gamma z) \approx \alpha\beta\gamma\mathfrak{q}(x, y, z) \ . \qquad \text{(Inv)}$$

Theorem 15 (Description of minimal minorities [2]). *Let $G \curvearrowright B$ be a Boolean group acting freely on B with s many orbits.*

- *There is a G-invariant Boolean Steiner 3-quasigroups if and only if $s = 2^n$ for some $n \in \mathbb{N}$, or is infinite;*
- *All G-invariant Boolean Steiner 3-quasigroups are minimal above $\overline{\langle G\rangle}$;*
- *For $s = 2^n$, the number of G-invariant Boolean Steiner 3-quasigroups is 1 for $n = 0$, and for $n \geq 1$ it is*

$$\frac{(2^n - 1)!|G|^{(2^n - n - 1)}}{\prod_{k=0}^{n-1}(2^n - 2^k)} \ .$$

Theorem 16 (Pálfy's Theorem for orbit-semiprojections [2]). *Let $G \curvearrowright B$ with s-many orbits and B finite or of the form $\mathrm{Aut}(\mathbb{B}) \curvearrowright B$ for $\mathbb{B}$ ω-categorical in a finite relational language. Then, for all $2 \leq k \leq s$, there is a k-ary orbit-semiprojection minimal above $\overline{\langle G\rangle}$.*

Our proof strategy for the main theorem: first classify almost minimal operations above $\overline{\langle G\rangle}$.
We use a weak version of Theorem 8 for almost minimal operations above a monoid (implicit in [1]) and show that certain behaviours cannot be witnessed by almost minimal operations since they give rise to non-injective unary operations.

Theorem 17 (Three types theorem [2]). *Let $G \curvearrowright B$ be such that G is not a Boolean group acting freely on B. Let s be the (possibly infinite) number of orbits of G on B. Let f be an **almost minimal operation** above $\overline{\langle G\rangle}$. Then, f is of one of the following types:*

1. *a unary operation;*
2. *a binary operation;*
3. *a k-ary orbit-semiprojection for $3 \leq k \leq s$.*

References

[1] Manuel Bodirsky and Hubie Chen. "Oligomorphic clones". In: Algebra Universalis 57 (2007), pp. 109–125.

[2] Paolo Marimon and Michael Pinsker. Minimal operations over permutation groups. 2024. arXiv: 2410.22060 [math.RA]. URL: https://arxiv.org/abs/2410.22060.

[3] Ivo G Rosenberg. "Minimal clones I: the five types". In: Lectures in universal algebra. Elsevier, 1986, pp. 405–427.