Minimal and intrinsic topologies on monoids of elementary embeddings

Paolo Marimon
joint work with J. de la Nuez Gonzalez, Zaniar Ghadernezhad, and Michael Pinsker

University of Oxford

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Outline

1. Preliminaries
2. The Zariski topology
3. Proving minimality

$\mathrm{Aut}(\mathbb{A})$ and $\mathrm{EEmb}(\mathbb{A})$.

$\mathbb{A}$:= a (first-order) structure.

We are interested in two natural spaces of symmetries:

- $\mathrm{Aut}(\mathbb{A})$:= automorphism group of $\mathbb{A}$;
- $\mathrm{EEmb}(\mathbb{A})$:= monoid of elementary embeddings of $\mathbb{A}$ ($f : \mathbb{A} \to \mathbb{A}$ preserving the truth of first-order formulas).

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Examples

- $\mathbb{A} := (\mathbb{N}; =)$.
  - $\mathrm{Aut}(\mathbb{A}) := \mathrm{Sym}(\mathbb{N})$;
  - $\mathrm{EEmb}(\mathbb{A}) := \mathrm{Inj}(\mathbb{N})$.
- $\mathbb{A} := (\mathbb{F}_q^{\aleph_0}; 0; +; (\lambda_r)_{r\in\mathbb{F}_q})$ ($\aleph_0$-dimensional vector space over $\mathbb{F}_q$).
  - $\mathrm{Aut}(\mathbb{A}) := \mathrm{GL}(\aleph_0, \mathbb{F}_q)$;
  - $\mathrm{EEmb}(\mathbb{A}) :=$ injective linear transformations $T : \mathbb{F}_q^{\aleph_0} \to \mathbb{F}_q^{\aleph_0}$.

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Topologies on spaces of symmetries

We can endow $S \in \{\mathrm{Aut}(\mathbb{A}), \mathrm{EEmb}(\mathbb{A})\}$ with the **topology of pointwise convergence $\boldsymbol{\tau_{\mathrm{pw}}}$**:

$$\text{for } a, b \in \mathbb{A},\ \mathcal{U}_{a,b} := \{f \in S |\ f(a) = b\}\ ,$$

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When $\mathbb{A}$ is countable, with τ_{pw}:

- $\mathrm{Aut}(\mathbb{A})$ is a Polish* topological group;
- $\mathrm{EEmb}(\mathbb{A})$ is a Polish topological monoid.

*__Polish:__ separable and completely metrizable.

ω-categorical structures

Definition (ω-categoricity)

$|\Omega| = \aleph_0$. $G \curvearrowright \Omega$ is **oligomorphic** if for all $n \in \mathbb{N}$, $G \curvearrowright \Omega^n$ has finitely many orbits in its diagonal action
$g \circ (a_1, \ldots, a_n) = (ga_1, \ldots, ga_n)$.
$\mathbb{A}$ is **ω-categorical** if $\mathrm{Aut}(\mathbb{A})$ is oligomorphic.

Examples

- $(\mathbb{N}; =)$;
- $(\mathbb{Q}; <)$;
- the countable random graph $\mathcal{R}$;
- countably infinite vector spaces over finite fields.

- In this context, $\mathrm{EEmb}(\mathbb{A})$ is the τ_{pw}-closure of $\mathrm{Aut}(\mathbb{A})$ in $\mathbb{A}^{\mathbb{A}}$.
- So, from now on, write $G := \mathrm{Aut}(\mathbb{A})$ and $\overline{G} := \mathrm{EEmb}(\mathbb{A})$.

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Why care about topology?

Let $S \in \{\mathrm{Aut}(\mathbb{A}), \mathrm{EEmb}(\mathbb{A})\}$ for $\mathbb{A}$ ω-categorical.

- S with τ_{pw} encodes $\mathbb{A}$ up to bi-interpretability (Ahlbrandt and Ziegler 1986; Lascar 1991); ▸ More on bi-interpretations
- If we can distinguish τ_{pw} amongst the topologies compatible with the algebraic structure of S, S encodes $\mathbb{A}$ as a group/monoid!
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- $(\mathbb{N}; =)$ (Gaughan 1967);
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Examples (for which τ_{pw} is NOT minimal on $\mathrm{Aut}(\mathbb{A})$)

- $(\mathbb{Q};<)$ (Chang and Gartside 2017): we can define a coarser **interval topology** τ_{int} with, for $a, b < c \in \mathbb{Q}$:

$$\mathcal{U}_{a,(b,c)} := \{f \in \mathrm{Aut}(\mathbb{Q}) |\ f(a) \in (b,c)\}\ ,$$

forming a sub-basis of open sets.

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Examples (for which τ_{pw} is NOT minimal on $\mathrm{Aut}(\mathbb{A})$)

- the **random bipartite graph** $\mathcal{B}$: we can define a coarser **localised topology** τ_{pw}^{X} (where X is one of the $\mathrm{Aut}(\mathcal{B})$-orbits) with for $a, b \in X$:

$$\mathcal{U}_{a,b} := \{f \in \mathrm{Aut}(\mathcal{B}) |\ f(a) = b\}\ ,$$

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Question

Are there any oligomorphic permutation groups for which τ_{pw} is not minimal on $\overline{G}$?†

†cf. Marimon and Pinsker 2025; Pinsker and Schindler 2023; Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023.

Algebraic closure

Definition (Algebraic closure)

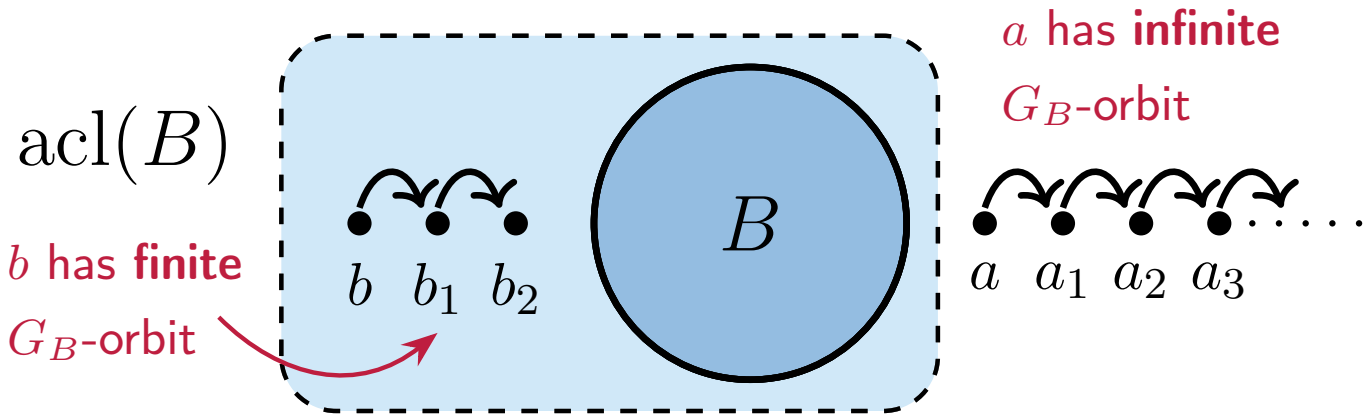
$G \curvearrowright \Omega$. For $B \subseteq \Omega$ finite,

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$$\mathrm{acl}(C) = \bigcup_{B \subseteq C \text{ finite}} \mathrm{acl}(B)\,.$$

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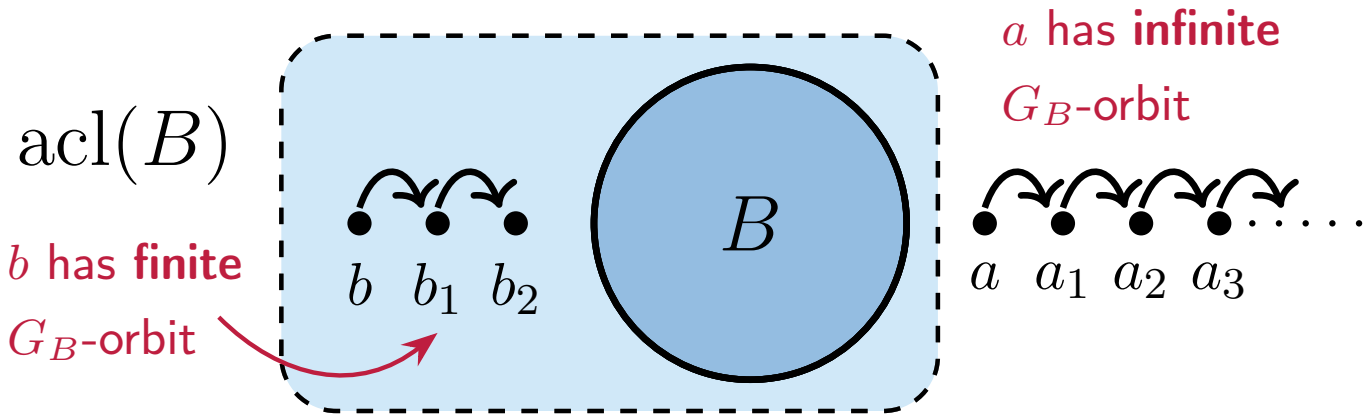
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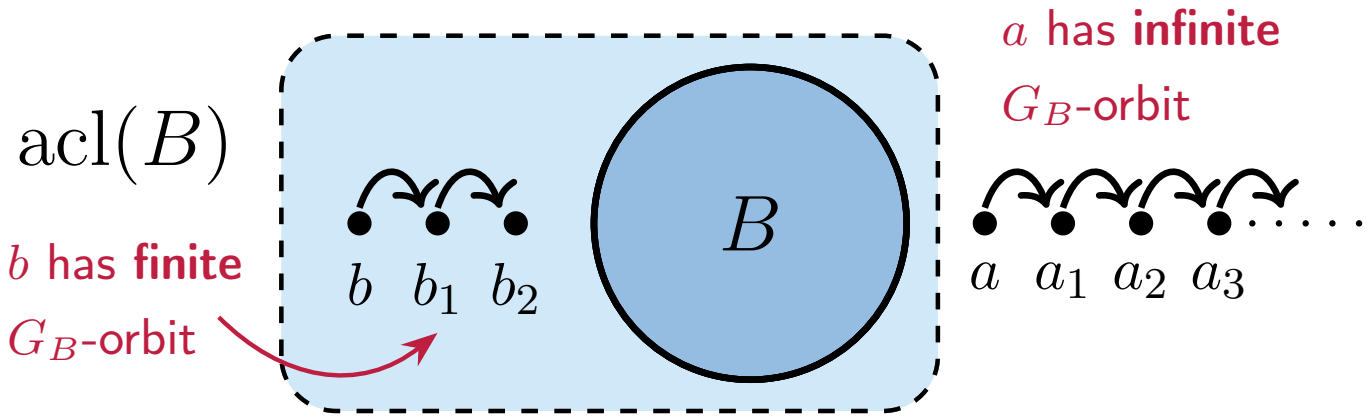
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Examples

- with no algebraicity: $(\mathbb{N}; =)$, $(\mathbb{Q}; <)$, $\mathcal{R}$, and $\mathcal{B}$;
- with algebraicity: vector spaces, and $\aleph_0$ disjoint copies of K_2.

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For $\mathbb{A}$ ω-categorical acl is **locally finite**: for finite B, $\mathrm{acl}(B)$ is finite.

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Theorem (Pinsker and Schindler 2023[‡])

Let $\mathbb{A}$ be ω-categorical and with no algebraicity.
Then, τ_{pw} is minimal on $\mathrm{EEmb}(\mathbb{A})$.

There are predecessors in Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023 and Elliott, Jonušas, Mitchell, Peresse, and Pinsker 2023.

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- This is very different from the case of $\mathrm{Aut}(\mathbb{A})$!

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Theorem (Pinsker and Schindler 2023)

Let $\mathbb{A}$ be ω-categorical and with no algebraicity.
*Then, τ_{pw} is minimal on $\mathrm{EEmb}(\mathbb{A})$ because it coincides with the **semigroup Zariski topology** τ_{Z}.*

The Zariski topology

Definition (the Zariski topology)

$\mathcal{S} :=$ a semigroup. The (semigroup) **Zariski** topology τ_{Z} has a sub-basis of open sets given by solutions to semigroup disequalities:

$$\{s \in \mathcal{S} \,|t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{l-1} s \dots q_1 s q_0\} \,,$$

for $k, l \geq 1$ and $t_0, \dots, t_k, q_0, \dots q_l \in \mathcal{S}$.

τ_{Z} is contained in any Hausdorff semigroup topology on $\mathcal{S}$.
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When is $\tau_{\mathrm{Z}} \subsetneq \tau_{\mathrm{pw}}$?

Pinsker and Schindler 2023 show that for $\mathbb{A} := \aleph_0$-many copies of K_2, $\tau_{\mathrm{Z}} \subsetneq \tau_{\mathrm{pw}}$ on $\mathrm{EEmb}(\mathbb{A})$ answering a question of Elliott, Jonušas, Mesyan, Mitchell, Morayne, and Peresse 2023.

We prove the following more general statement:

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We prove the following more general statement:

Theorem (de la Nuez Gonzalez, Ghadernezhad, M., Pinsker 2026)

Let $G \curvearrowright \Omega$ have locally finite algebraic closure and non-trivial centre. Then, the semigroup Zariski topology τ_{Z} on $\overline{G}$ is not Hausdorff.

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Proof sketch

Proof sketch of the theorem.

A semigroup disequality $w(s)$

$$t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{k-1} s \dots q_1 s q_0$$

is **singular** if $k = l$ and **non-singular** otherwise.

Write $\mathcal{O}_{w(s)}$ for the τ_{Z}-open set of solutions to $w(s)$.

- if $w(s)$ is singular, for $\gamma \in Z(G)$, $1 \in \mathcal{O}_{w(s)}$ if and only if $\gamma \in \mathcal{O}_{w(s)}$;
- if $w(s)$ is non-singular, then $\mathcal{O}_{w(s)}$ is dense in $\overline{G}$ (wrt τ_{pw}) (use Neumann's Lemma).
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- if $w(s)$ is non-singular, then $\mathcal{O}_{w(s)}$ is dense in $\overline{G}$ (wrt τ_{pw}) (use Neumann's Lemma).
- Because of this any $\gamma \in Z(G)$ cannot be separated from 1 by τ_{Z}-open sets (in neither G nor $\overline{G}$).



Proof sketch

Proof sketch of the theorem.

A semigroup disequality $w(s)$

$$t_k s t_{k-1} s \dots t_1 s t_0 \neq q_l s q_{k-1} s \dots q_1 s q_0$$

is **singular** if $k = l$ and **non-singular** otherwise.
Write $\mathcal{O}_{w(s)}$ for the τ_{Z}-open set of solutions to $w(s)$.

- if $w(s)$ is singular, for $\gamma \in Z(G)$, $1 \in \mathcal{O}_{w(s)}$ if and only if $\gamma \in \mathcal{O}_{w(s)}$;
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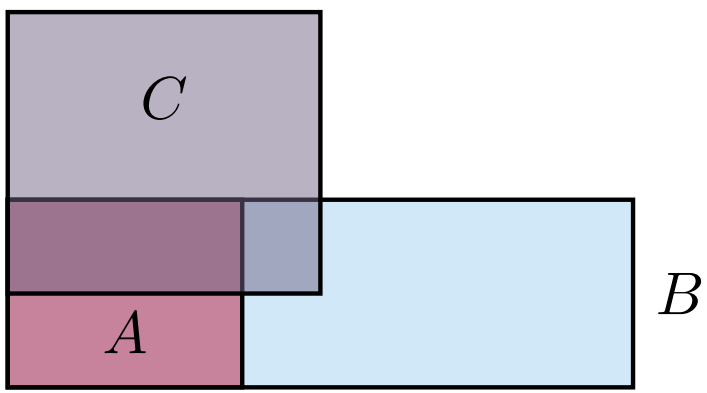
More generally: the standard strategy of $\tau_{\mathrm{pw}} = \tau_{\mathrm{Z}}$ easily fails the moment we have algebraicity.
Can we still prove minimality by other means?

Modularity

Definition (Modularity)

$G \curvearrowright \Omega$ with locally finite acl.
acl is **modular** if for finite algebraically closed A, B, C s.t. $A \subseteq B$,

$$\mathrm{acl}(A \cup (C \cap B)) = \mathrm{acl}(A \cup C) \cap B \ .$$

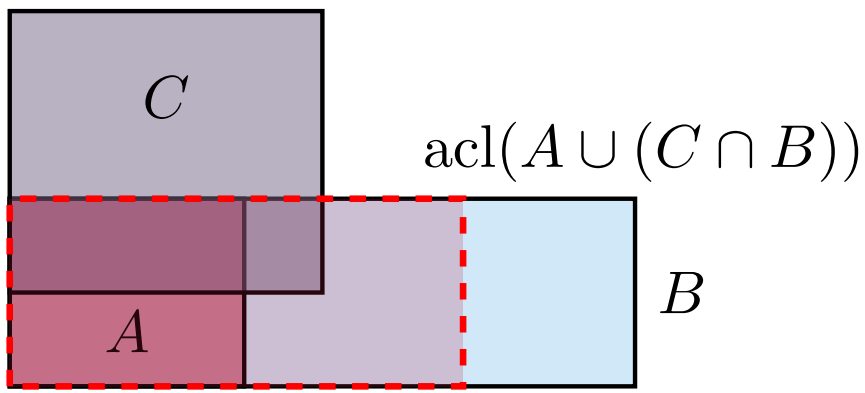


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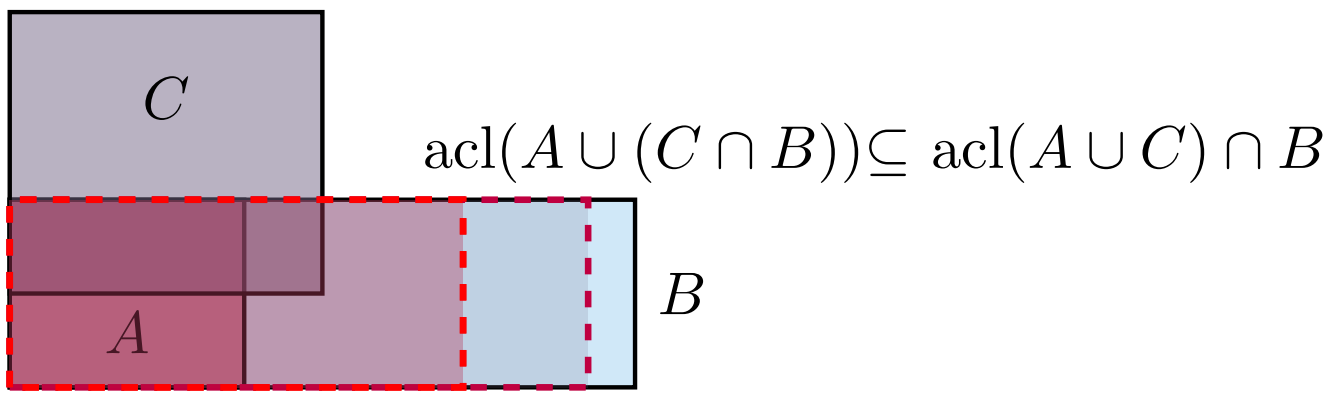


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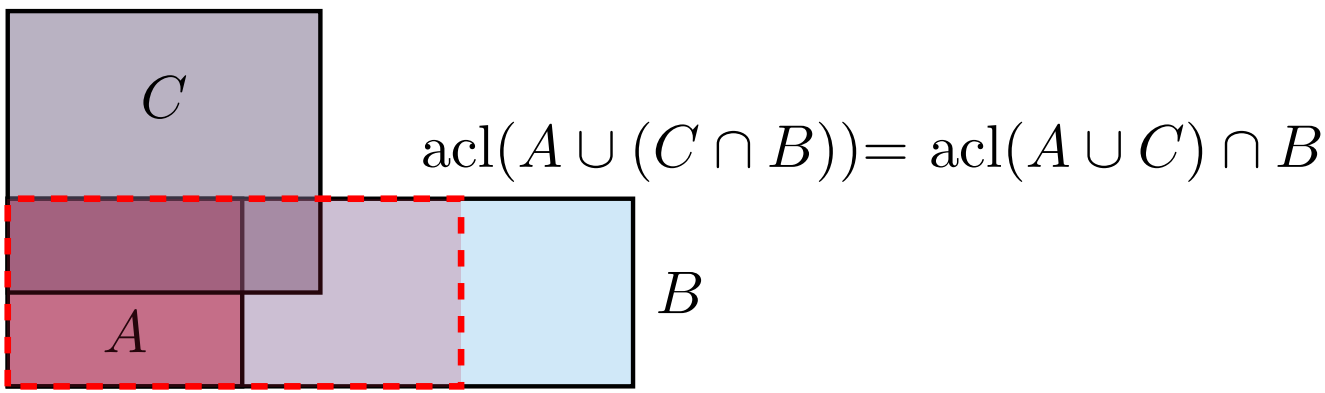


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Examples

- whenever $\mathbb{A}$ has no algebraicity, acl is modular;
- vector spaces or projective spaces over finite fields (possibly with forms, or densely ordered).
- sometimes one needs to be more subtle (affine space over $\mathbb{F}_q$): there is $\mathbb{A}'$ with $\mathrm{EEmb}(\mathbb{A}) = \mathrm{EEmb}(\mathbb{A}')$ (with τ_{pw}) and acl is modular in $\mathbb{A}'$ (but not in $\mathbb{A}$).

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Morally: acl is modular when it is
"not more complicated than in a vector space".

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Theorem (de la Nuez Gonzalez, Ghadernezhad, M., & Pinsker 2026)

Let $G \curvearrowright \Omega$ be oligomorphic and such that acl is modular.
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This subsumes Pinsker and Schindler 2023.

Rigidity and dispersivity

$G \curvearrowright \Omega$. Consider a semigroup topology $\tau \subseteq \tau_{\mathrm{pw}}$ on $\overline{G}$.

Definitions (Rigidity and dispersivity)

$t \in \overline{G}$ is **τ-rigid** on $a \in \Omega$ if there is some τ-neighbourhood $\mathcal{V}$ of t such that, for all $s \in \mathcal{V}$, $s(a) \in \mathrm{acl}(t(a))$.

$t \in \overline{G}$ is **τ-dispersive** on $a \in \Omega$ if for every finite $B \subseteq \Omega$ and any τ-neighbourhood $\mathcal{V}$ of t there is $s \in \mathcal{V}$ such that $a \notin s(B)$.

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Lemma (de la Nuez Gonzalez, Ghadernezhad, M., and Pinsker 2026)

$G \curvearrowright \Omega$ with locally finite acl. Suppose that $\tau \subsetneq \tau_{\mathrm{pw}}$ is a semigroup topology and every $t \in \overline{G}$ is τ-rigid on Ω. Then, τ is not Hausdorff.

Proof idea of the main theorem

Proof idea.

Suppose $\tau \subsetneq \tau_{\mathrm{pw}}$ is a Hausdorff semigroup topology. Consider

$$\Delta := \{a \in \Omega | \text{ every } s \in \overline{G} \text{ is } \tau\text{-rigid on } a\} .$$

By previous Lemma, $\Delta \subsetneq \Omega$.
We can find $t \in \overline{G}$ satisfying a strong form of τ-dispersivity on $\Omega \setminus \Delta$.
By **modularity**, find a "generic" copy $\mathbb{A}^\star$ of $\mathbb{A}$ within itself.
In particular,

$$\mathcal{U} := \{s \in \overline{G} | \operatorname{Im}(s) \subsetneq \mathbb{A}^\star\}$$

is τ_{Z}-open (and so τ-open) and a neighbourhood of t.
We exploit τ-dispersivity and genericity of $\mathbb{A}^\star$ to find:
$u \in \mathcal{U}$ such that $\operatorname{Im}(u) \subseteq \mathbb{A}^\star$. (CONTRADICTION!)

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Thank you!

For $\mathbb{A}$ ω-categorical

- We show τ_{Z} is not Hausdorff on $\mathrm{EEmb}(\mathbb{A})$ whenever $\mathrm{Aut}(\mathbb{A})$ has non-trivial centre;
- We find new ways to prove minimality of τ_{pw} as long as acl is sufficiently tame (i.e., modular).

Survey:

Paper:

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Bi-interpretations

Definition

An **interpretation** of $\mathbb{B}$ in $\mathbb{A}$ is a partial surjection $I : A^d \to B$ for some $d \in \mathbb{N}$ such that for each relation R of $\mathbb{B}$ defined by an atomic formula, $I^{-1}(R)$ is definable in $\mathbb{A}$ (without parameters).

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Example

$(\mathbb{C}; 0; 1; \times; +)$ is interpretable in $(\mathbb{R}; 0; 1; \times; +)$ (by the Argand plane).

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Interpretations compose naturally.
If I is an interpretation of $\mathbb{B}$ in $\mathbb{A}$ and J is an interpretation if $\mathbb{A}$ in $\mathbb{B}$, $\mathbb{A}$ and $\mathbb{B}$ are **bi-interpretable** if $I \circ J$ and $J \circ I$ are definable in $\mathbb{B}$ and $\mathbb{A}$ respectively.

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Example

$(\mathbb{N}; =)$ is bi-interpretable with the **Johnson graph** $\mathfrak{J}(2)$:

- Domain: $[\mathbb{N}]^2$ (unordered pairs);
- Edges $E := \{(\{a, b\}, \{c, d\}) \in ([\mathbb{N}]^2)^2 |\ |\{a, b\} \cap \{c, d\}| = 1\}$.

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Theorem (Coquand in Ahlbrandt and Ziegler 1986, cf. Lascar 1991)

Let $\mathbb{A}$ and $\mathbb{B}$ be ω-categorical structures. Then, TFAE:

- *$\mathbb{A}$ and $\mathbb{B}$ are bi-interpretable;*
- *$\mathrm{Aut}(\mathbb{A})$ and $\mathrm{Aut}(\mathbb{B})$ are isomorphic as topological groups;*
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Lovely pairs

Definition (Lovely pair[§])

$\mathbb{A}$ ω-categorical. $L \subseteq \mathbb{A}$ algebraically closed is a **lovely pair** if

- (coheir) for finite algebraically closed A, B with $\operatorname{acl}(B) \cap \operatorname{acl}((L \cap B) \cup A) = \operatorname{acl}(L \cap B)$, there is $g \in G_B$ such that $g(A) \subseteq L$;
- (extension) for finite algebraically closed A, B there is $g \in G_B$ with $\operatorname{acl}(g(A) \cup B) \cap \operatorname{acl}(L \cup B) = B$.

[§]Berenstein and Vassiliev 2010 (cf. Ben-Yaacov, Pillay, and Vassiliev 2003; Poizat 1983)

Lovely pairs

Definition (Lovely pair[§])

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Examples

- in $(\mathbb{N}; =)$ this is an infinite-coinfinite set;
- in $(\mathbb{Q}; <)$ this is a dense codense set;
- in vector space $\mathbb{F}_q^{\aleph_0}$ this is an infinite-dimensional subspace of infinite codimension.

[§]Berenstein and Vassiliev 2010 (cf. Ben-Yaacov, Pillay, and Vassiliev 2003; Poizat 1983)

What is left out?

- Droste's almost homogeneous semilinear orders
  (but for these we can prove $\tau_{\mathrm{pw}} = \tau_{\mathrm{Z}}$);
- Hrushovski constructions
  (but these seem to be amenable to a variant of our proof);
- atomless Boolean algebras
  (τ_{pw} should be minimal by other methods).