Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $(\mathcal{A}_i \mid i \in I)$ be a family of $\mathcal{L}$-structures, and let $\mathcal{F}$ be an ultrafilter on I. Let $\sim$ be the equivalence relation on $\prod_{i\in I} A_i$, where for $(a_i)_{i\in I}, (b_i)_{i\in I} \in \prod_{i\in I} A_i$ we have that

$$(a_i)_{i\in I} \sim (b_i)_{i\in I} \text{ if and only if } \{i \in I \mid a_i = b_i\} \in \mathcal{F}.$$

We define the **ultraproduct** $\mathcal{U} := \prod_{i\in I} \mathcal{A}_i/\mathcal{F}$ as follows:

- The domain U of $\mathcal{U}$ is the set of all equivalence classes $(\prod_{i\in I} A_i)/\sim$;
- For c a constant $\mathcal{L}$-symbol, $c^{\mathcal{U}} := [(c^{\mathcal{A}_i})_{i\in I}]_\sim$;
- For R an n-ary relation symbol in $\mathcal{L}$, and $[(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim \in U$,

$$R^{\mathcal{U}}([(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim) \text{ if and only if } \{i \in I \mid R^{\mathcal{A}_i}(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} .$$

- For f an n-ary function symbol in $\mathcal{L}$ and $[(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim \in U$,

$$f^{\mathcal{U}}([(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim) := [(f^{\mathcal{A}_i}(a_i^1, \ldots, a_i^n))_{i\in I}]_\sim .$$

EXERCISE 1. Show that $\mathcal{U}$ is well-defined.

EXERCISE 2. Prove Łoś' Theorem:
For any $\mathcal{L}$-formula, $\phi(x_1, \ldots, x_n)$ and any $[(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim \in U$,

$$\mathcal{U} \models \phi([(a_i^1)_{i\in I}]_\sim, \ldots, [(a_i^n)_{i\in I}]_\sim) \text{ if and only if } \{i \in I \mid \mathcal{A}_i \models \phi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} .$$

EXERCISE 3. Deduce the compactness theorem from Łoś' Theorem. That is, show that a theory T is consistent (i.e. it has a model) if and only if every finite subset $S \subseteq T$ is consistent.

Definition 2. An $\mathcal{L}$-structure $\mathcal{M}$ is **pseudofinite** if for every $\mathcal{L}$-sentence σ such that $\mathcal{M} \models \sigma$ there is a finite $\mathcal{L}$-structure $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \sigma$.

$\star\star$ **EXERCISE 4.** Show that an $\mathcal{L}$-structure $\mathcal{M}$ is pseudofinite if and only if it is elementarily equivalent to an ultraproduct of finite $\mathcal{L}$-structures, i.e. there is an ultraproduct $\mathcal{U}$ of finite $\mathcal{L}$-structures such that $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{U})$.

EXERCISE 5. A graph is a structure with a binary relation which is symmetric and irreflexive. A graph is **connected** if there is a (finite) path between any two of its vertices. Prove that the class of connected graphs is not **elementary**, that is, there is no first order theory whose models are exactly the connected graphs.

Definition 3. A **well-order** is a linear order such that every non-empty set has a least element.

EXERCISE 6. Let $\phi(x, y)$ be an $\mathcal{L}$-formula. Suppose that $\mathcal{A}$ is an infinite $\mathcal{L}$-structure where $\phi(A^2)$ forms a well-order on A. Show that there is an $\mathcal{L}$-structure B with the same theory as A for which $\phi(B^2)$ is a linear ordering which is not a well-order.

$\star\star$ **EXERCISE 7.** Let G be an infinite simple group (i.e. the only normal subgroups of G are the trivial group and itself). Show that for every infinite cardinality $\lambda \leq |G|$, G has a simple subgroup of cardinality λ.

1 Hints

Spoilers ahead! Keep in mind that for some of the exercises it should be sufficient for you to remind yourself of some of the relevant definitions and theorems.

- EXERCISES 2, 3, 4: For the exercises on ultraproducts, you might have to remind yourself of some of the basic properties of ultrafilters. In particular, we say that $\mathcal{A} \subseteq \mathcal{P}(I)$ has the **finite intersection property** if any collection of finitely many sets from $\mathcal{A}$ has non-empty intersection. Any $\mathcal{A} \subseteq \mathcal{P}(I)$ with the **finite intersection property** is contained in a filter $\mathcal{F}$ on I. Then, you just need to remember the Ultrafilter Lemma: any filter on I extends to an ultrafilter on I.
- EXERCISE 6: Adapt the proof of the upwards Löwenheim-Skolem Theorem.
- EXERCISE 7: Apply the downwards Löwenheim-Skolem Theorem. Recall the definition of **normal subgroup generated by** a set of elements in a group:

  Definition 4. Let $A \subseteq G$. The **normal subgroup generated** by A in G is the smallest normal subgroup of G containing A. It is given by the subgroup of G generated by

  $$\bigcup_{g \in G} gAg^{-1}.$$

  If you struggle with proving this result, try and prove the following slightly easier result:

  EXERCISE 7′: Let G be an infinite connected graph. Show that for every infinite cardinality $\lambda \leq |G|$, G has a connected subgraph H elementarily equivalent to G and of cardinality λ.

  You know that being connected is not an elementary property. So how are you going to get that H is still connected?