Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Fact 1. *Let G and H be groups. Then,* $\mathrm{Aut}(G \times H)$ *has a subgroup isomorphic to* $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$.

EXERCISE 1. Let G and H be countable ω-categorical groups. Show that $G \times H$ is still ω-categorical.

Fact 2. *An Abelian group of finite exponent is a direct sum of finite cyclic groups.*

EXERCISE 2. Let $\mathcal{A}_n$ be the class of finite Abelian groups of exponent dividing n (that is, $A \in \mathcal{A}_n$ if and only if $nA = 0$). Show that $\mathcal{A}_n$ is a Fraïssé class and that its Fraïssé limit is isomorphic to $\mathbb{Z}_n^{(\omega)}$, i.e. the direct sum of countably many copies of the cyclic group of order n. [Hint: for the amalgamation property: given $f_i : A \to B_i$ in $\mathcal{A}_n$. Let $D = B_0 \oplus B_1$ and $E := \{(f_0(a), -f_1(a)) | a \in A\}$ and consider D/E as the amalgam.]

Given $n \in \mathbb{N}$, note that the number of isomorphism types of n-generated substructures in $\mathcal{A}_n$ is finite. Deduce that $\mathbb{Z}_n^{(\omega)}$ is ω-categorical.

Deduce that an infinite Abelian group is ω-categorical if and only if it has finite exponent.

Fact 3 (Neumann's Lemma)**.** *Suppose G is a group acting on a set X and all G-orbits on X are infinite. Let $B, C \subseteq X$ be finite. Then there is some $g \in G$ such that $B \cap gC = \emptyset$.*

Definition 4. We say that an $\mathcal{L}$-structure M has trivial algebraic closure if $\mathrm{acl}(A) = A$ for all finite sets.

EXERCISE 3. Let $\mathcal{C}$ be a Fraïssé class in a finite relational language and write $\mathcal{M}$ for the resulting Fraïssé limit. Then, $\mathcal{M}$ has trivial algebraic closure if and only if $\mathcal{C}$ has the **strong amalgamation property**:

- For $A, B_0, B_1 \in \mathcal{C}$ and embeddings $f_i : A \to B_i$ for $i \in \{0, 1\}$ there is a $C \in \mathcal{C}$ and embeddings $g_i : B_i \to C$ for $i \in \{0, 1\}$ such that $g_0 \circ f_0 = g_1 \circ f_1$ and $g_0(A_0) \cap g_1(A_1) = g_0(f_0(A))$.

[Hint: For the '$\Leftarrow$' implication, consider a finite set $A \subseteq M$ and $b \notin A$. You need to show that the $\mathrm{Aut}(M/A)$-orbit of b is infinite, where $\mathrm{Aut}(M/A)$ is the stabilizer of A. For the '$\Rightarrow$' implication you need Neumann's Lemma.]
Give an example of a countable homogeneous relational structure which does not have trivial algebraic closure.

EXERCISE 4. Prove Ramsey's Theorem:

Theorem 5 (Ramsey's Theorem, infinite version)**.** *Let A be an infinite set and $n \in \omega$. Partition the set of n-element subsets $[A]^n$ into subsets $C_1, \ldots, C_k$. Then, there is an infinite subset of A whose n-element subsets all belong to the same subset C_i.*

[Hint: Prove the theorem first for $n = 1, 2$ and then think of how the induction works]

Definition 6. Let G act on X transitively. We say that the action of G on X is **primitive** if there are no non-trivial G-invariant equivalence relations on X. Otherwise, we say the action is **imprimitive**.
An **orbital** Δ of G is an orbit of G on X^2. For each $a \in X$, the set $\{(a, a) | a \in X\}$ is called the **diagonal orbital**. For an orbital Δ of G we define the **orbital graph** of G with respect to Δ to be the directed graph that has X as vertex set and Δ as its set of directed edges.

Fact 7 (Higman's Theorem)**.** *Let G act transitively on X. Then, the action of G on X is primitive if and only if for every orbital Δ of G except for the diagonal orbital, the orbital graph of Δ is connected.*

$\star\star$ **EXERCISE 5.** Show that if a countable homogeneous graph is disconnected, then each connected component must be a complete graph.

Classify the countably infinite homogeneous graphs with imprimitive automorphism group (without using the Lachlan & Woodrow classification of homogeneous graphs).