Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. A class $\mathcal{C}$ of $\mathcal{L}$-structures is said to be **finitely axiomatizable** if it is the class of models of an $\mathcal{L}$- theory consisting of finitely many $\mathcal{L}$-sentences.

Definition 2. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that $\mathcal{C}$ is an **elementary class** if there is an $\mathcal{L}$-theory whose models are exactly the structures in $\mathcal{C}$.

EXERCISE 1. Show that a class $\mathcal{C}$ of $\mathcal{L}$-structures is finitely axiomatizable if and only if both $\mathcal{C}$ and its complement (i.e. the class of those $\mathcal{L}$-structures that are not in $\mathcal{C}$) are elementary.

Notation 3. We write $\mathcal{A} \preceq \mathcal{M}$ to say that $\mathcal{A}$ is elementarily embedded in $\mathcal{M}$.

Definition 4. Let I be a linearly ordered set. A family of structures $(\mathcal{A}_i)_{i\in I}$ forms an **elementary chain** if $\mathcal{A}_i \preceq \mathcal{A}_j$ for $i < j$ from I.

EXERCISE 2. Prove Tarski's Chain Lemma: let $(\mathcal{A}_i)_{i\in I}$ be an elementary chain. Consider the structure

$$\mathcal{A} := \bigcup_{i\in I} \mathcal{A}_i.$$

Then, $\mathcal{A}$ is an elementary extension of each $\mathcal{A}_i$.

EXERCISE 3. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. By $\mathrm{Th}(\mathcal{C})$ we denote the common theory of all $\mathcal{L}$-structures in $\mathcal{C}$. That is,

$$\mathrm{Th}(\mathcal{C}) := \{\phi \in \mathrm{Sen}(\mathcal{L}) | \mathcal{A} \vDash \phi \text{ for all } \mathcal{A} \in \mathcal{C}\}.$$

1. Show that an $\mathcal{L}$-structure $\mathcal{M}$ is a model of $\mathrm{Th}(\mathcal{C})$ if and only if it is elementarily equivalent to an ultraproduct of elements of $\mathcal{C}$;
2. Show that $\mathcal{C}$ is an elementary class if and only if $\mathcal{C}$ is closed under ultraproducts and elementary equivalence.

EXERCISE 4. Let $\mathcal{L}_{\mathrm{or}} = (<, +, -, \cdot, 0, 1)$ be the language of ordered rings. Let $\mathcal{L}_{\mathrm{or}f} = \mathcal{L}_{\mathrm{or}} \cup \{f\}$, where f is a unary function symbol. Let $\mathcal{R} := (\mathbb{R}; <, +, -, \cdot, 0, 1, f)$ be an $\mathcal{L}_{\mathrm{or}f}$-structure where $(\mathbb{R}; <, +, -, \cdot, 0, 1)$ is the ordered field of real numbers and $f^{\mathcal{R}} : \mathbb{R} \to \mathbb{R}$ is a unary function with $f^{\mathcal{R}}(0) = 0$.

1. Show there is an elementary extension $\mathcal{A} \succeq \mathcal{R}$ containing an infinitesimal, that is, an element $\epsilon \neq 0$ such that for all $n \in \mathbb{N}$,
$$\mathcal{A} \vDash n|\epsilon| < 1;$$
2. Assume that $f^{\mathcal{R}}$ is continuous at 0. Show that for any elementary extension $\mathcal{A} \succeq \mathcal{R}$ and any infinitesimal $\epsilon \in A$, $f^{\mathcal{A}}(\epsilon)$ is also infinitesimal;
3. Suppose that for all $\mathcal{A} \succeq \mathcal{R}$ and infinitesimal $\epsilon \in \mathcal{A}$, $f^{\mathcal{A}}(\epsilon)$ is also infinitesimal. Show that $f^{\mathcal{R}}$ is continuous at 0.

Definition 5. We say that $\mathcal{M}$ is **ω-saturated** if all types in finitely many variables over finite subsets of M are realised in $\mathcal{M}$.

$\star\star$ **EXERCISE 5.** Let $\mathcal{L}_{ar} := (+, \times)$ be the language of arithmetics. Let $(\mathbb{N}; +, \times)$ be the natural numbers, where each $\mathcal{L}_{ar}$-symbol is given its standard interpretation. We call a model of $\mathrm{Th}(\mathbb{N})$ which is not isomorphic to $\mathbb{N}$ a **non-standard model of arithmetics**.

1. Prove that there is a countable non-standard model of arithmetics $\mathcal{N} \succeq \mathbb{N}$ containing an element n^* which has infinitely many distinct prime factors;
2. Show that there are uncountably many non-isomorphic countable models of $\mathrm{Th}(\mathbb{N})$ [Hint: the idea from the first part of this question should help!]. Deduce there is no countable ω-saturated model for $\mathrm{Th}(\mathbb{N})$.

Notation 6. Let a, b be n-tuples from $\mathcal{M}$ and $C \subseteq \mathcal{M}$. We write $a \equiv_C b$ to say that the type of a over C is the same as the type of b over C, that is $\mathrm{tp}(a/C) = \mathrm{tp}(b/C)$. We write $a \equiv b$ to say that a and b have the same type over $\emptyset$.

$\star\star$ **EXERCISE 6.** Let $\mathcal{L}$ be a relational language. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let a, b in $\mathcal{M}$ be such that $a \equiv b$. We want to prove that there is $\mathcal{N} \succeq \mathcal{M}$ and an automorphism σ of $\mathcal{N}$ such that $\sigma a = b$. This exercise guides you through a proof of this:

1. Show that it is sufficient to prove that for $\mathcal{L}$ a *finite* relational language and $\mathcal{M}$ a *countable* $\mathcal{L}$-structure with $a, b \in M$ such that $a \equiv b$ there is *countable* $\mathcal{N} \succeq \mathcal{M}$ and an automorphism σ of $\mathcal{N}$ such that $\sigma a = b$. Hence, we will assume this in the following tasks;

Consider the language $\mathcal{L}'$ consisting of $\mathcal{L}(M)$, that is, $\mathcal{L}$ together with constant symbols for all elements of M and $2n$-ary relations $E_n(x_1, \ldots, x_n, y_1, \ldots, y_n)$ for each $n \in \mathbb{N}$. Let $\mathcal{M}_M$ be the $\mathcal{L}(M)$-expansion of M, where each element of M is named by a constant. Let $\mathrm{Th}(M_M)$ be the theory of $\mathcal{M}_M$. Let Γ be the $\mathcal{L}'$-theory consisting of $\mathrm{Th}(M_M)$ adjoined with axioms saying that each E_n is an equivalence relation on n-tuples and

$$\forall \overline{x} \forall \overline{y} \forall u \exists v (E_n(\overline{x}, \overline{y}) \to E_{n+1}(\overline{x}u, \overline{y}v)) \tag{1}$$

$$\forall \overline{x} \forall \overline{y} (E_n(\overline{x}, \overline{y}) \to (\phi(\overline{x}, a) \leftrightarrow \phi(\overline{y}, b)) \tag{2}$$

for each n and each atomic $\mathcal{L}$-formula $\phi(\overline{x}, \overline{z})$.

2. Show that if Γ is satisfiable, then there is a countable model $\mathcal{N} \succeq \mathcal{M}$ and an automorphism σ of $\mathcal{N}$ such that $\sigma a = b$;

3. Show that Γ is finitely satisfiable. [Hint: start with $\Gamma' \subseteq \Gamma$ containing $\mathrm{Th}(M)$ and only mentioning E_1 and E_2. Try and interpret E_2 as the equivalence relation holding for all pairs which have the same quantifier-free type. How should you interpret E_1? How can the assumption that $\mathcal{L}$ is finite help you?]