Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $\mathcal{L}_{\text{ring}}$ be the language of rings. For p prime, we denote by ACF_p the theory of algebraically closed fields of characteristic p. Similarly, ACF_0 denotes the theory of algebraically closed fields of characteristic 0.

EXERCISE 1. Let ϕ be an $\mathcal{L}_{\text{ring}}$-sentence. Prove that the following are equivalent:

- $\text{ACF}_0 \models \phi$;
- for all sufficiently large primes p, $\text{ACF}_p \models \phi$;
- there are arbitrarily large primes p such that $\text{ACF}_p \models \phi$.

Deduce that ACF_0 is not finitely axiomatizable.

Definition 2. Let K be a field. We say that a map $f : K^n \to K^n$ is a **polynomial map** if it is of the form

$$f(x_1, \ldots, x_n) = (p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n)),$$

where $p_i \in K[x_1, \ldots, x_n]$ for each $i \leq n$.

The following theorem was first proven using model theory (indeed, you only need Exercise 1 and the fact that ACF_p is complete for each prime p):

$\star\star$ **EXERCISE 2.** Prove the Ax-Grothendieck Theorem: let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If f is one to one, then f is onto. [Hint: for $d \in \mathbb{N}$, there is an $\mathcal{L}_{\text{ring}}$-sentence Φ_d expressing that for all polynomial maps f such that every polynomial p_i in it has degree $\leq d$, if f is one-to-one, then it is onto.]

Definition 3. Let $\mathcal{L}_{\text{gr}}$ consist of a single binary relation E and T_{gr} be the theory of undirected graphs without loops. For $n, m \in \mathbb{N}$, the Alice restaurant axiom $A_{n,m}$ is the following $\mathcal{L}_{\text{gr}}$ sentence:

$$\forall x_1, \ldots, x_n, y_1, \ldots, y_m \Big(\bigwedge_{i,j} x_i \neq y_j \to (\exists z \bigwedge_{i \leq n} E(z, x_i) \wedge \bigwedge_{j \leq m} (\neg E(z, y_j) \wedge z \neq y_j)) \Big).$$

Let T_{rg} be obtained by $T_{\text{gr}} \cup \{A_{n,m} | n, m \in \mathbb{N}\}$. We call T_{rg} the **theory of the random graph**.

Definition 4. We say that an $\mathcal{L}$-formula ϕ is **quantifier-free** if it does not contain any quantifier.

Definition 5. We say that an $\mathcal{L}$-theory T has **quantifier elimination** if every $\mathcal{L}$-sentence is equivalent, modulo T, to a quantifier-free $\mathcal{L}$-formula. That is, for every $\mathcal{L}$-formula $\phi(\overline{x})$ with free variables $\overline{x}$, there is a quantifier-free $\mathcal{L}$-formula $\psi(\overline{x})$ such that

$$T \vdash \forall \overline{x} (\phi(\overline{x}) \leftrightarrow \psi(\overline{x})) \ .$$

EXERCISE 3. Show that the theory of the random graph is ω-categorical. Deduce that it is complete. Further prove that the theory of the random graph has quantifier elimination.

The following is really an exercise in probability, but it is the reason for the name of the random graph.

$\star\star$ **EXERCISE 4.** Let $0 < p < 1$. Take n vertices and for each pair of distinct vertices choose independently at random with probability p whether they form an edge. Let $G(n, p)$ be the graph obtained in this manner. Show that for each $k, l \in \mathbb{N}$,

$$\mathbb{P}(G(n, p) \models A_{k,l}) \to 1 \text{ as } n \to \infty.$$

Prove that for any $\mathcal{L}_{\text{gr}}$-sentence ϕ, $T_{rg} \models \phi$ if and only if

$$\mathbb{P}(G(n, p) \models \phi) \to 1 \text{ as } n \to \infty.$$

EXERCISE 5. Show that the following theories do NOT have quantifier elimination:

- $\mathrm{Th}(\mathbb{N}; <)$;
- $\mathrm{Th}(\mathbb{Z}; +)$;
- $\mathrm{Th}(\mathbb{R}; 0, 1, +, \cdot, -)$;
- $\mathrm{Th}(\mathbb{Q}; 0, 1, +, \cdot, -, <)$.