Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $\mathcal{M}$ be a $\mathcal{L}$-structure and $A \subset M$. The **substructure of** M **generated by** A is the smallest $\mathcal{L}$-substructure of $\mathcal{M}$ containing A. That is, the substructure of $\mathcal{M}$ with universe

$$\bigcap\{N | A \subseteq N, \mathcal{N} \subseteq \mathcal{M} \text{ is an } \mathcal{L}\text{-structure}\}.$$

We say that a substructure $\mathcal{N}$ of $\mathcal{M}$ is **finitely generated** if it is generated by a finite set.

Definition 2. Let $\mathcal{P}$ be a property of $\mathcal{L}$-structures. We say that $\mathcal{M}$ has $\mathcal{P}$ **locally** if $\mathcal{P}$ holds for all finitely generated substructures of $\mathcal{M}$.

EXERCISE 1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{C}$ an elementary class of $\mathcal{L}$-structures. Show that $\mathcal{M}$ is embeddable in a member of $\mathcal{C}$ if and only if $\mathcal{M}$ is locally embeddable in members of $\mathcal{C}$.
Deduce that every linear ordering can be embedded in a dense linear order.

Definition 3. An Abelian group $(G; +, -, 0)$ is **torsion-free** if for all $n \in \mathbb{N}$, the only element x satisfying $nx = 0$ is $x = 0$.
An Abelian group is **divisible** if for all n and all $g \in G$ there is $y \in G$ such that $ny = g$. We write DAG for the theory of non-trivial torsion-free divisible Abelian groups.

EXERCISE 2. Show that $\text{DAG}_\forall$ is the theory of torsion-free Abelian groups. [You may use EXERCISE 1 and basic facts about Abelian groups.]

EXERCISE 3. Show that the following definitions of ω-saturation are equivalent:

- for all $A \subset M$ finite and $p \in S_1(A)$, p is realised in M;
- for all $k \in \mathbb{N}$, $A \subset M$ finite and $p \in S_k(A)$, p is realised in M.

EXERCISE 4. Show that if two countable ω-saturated structures are elementarily equivalent then they are isomorphic.

Definition 4. We say that an $\mathcal{L}$-theory T is **strongly minimal** if for any $\mathcal{M} \models T$, every definable subset (with parameters) of M is either finite or cofinite.

EXERCISE 5. Let $(\mathbb{Z}; s)$ be the integers with a function symbol s denoting the successor function $s(x) = x + 1$. Prove that $\text{Th}(\mathbb{Z}; s)$ has quantifier elimination. Deduce that $\text{Th}(\mathbb{Z}; s)$ is strongly minimal.

$\star\star$ **EXERCISE 6.** Show that every definable subset (with parameters) of $(\mathbb{N}; <)$ is either finite or cofinite but that $\text{Th}(\mathbb{N}; <)$ is not strongly minimal. [Hint: you know that $\text{Th}(\mathbb{N}; <)$ does not have quantifier elimination, but it might be helpful to think about an expansion of $(\mathbb{N}; <)$ to some language for which quantifier elimination holds.]