Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. An $\mathcal{L}$-formula is **positive quantifier-free** if it is quantifier-free and containing no negations.

EXERCISE 1. Let T be a complete $\mathcal{L}$-theory and $\phi(\bar{x})$ be an $\mathcal{L}$-formula such that $T \models \exists\bar{x}\phi(\bar{x})$. Prove that the following are equivalent:

1. there is positive quantifier-free $\psi(\bar{x})$ such that
$$T \models \forall\bar{x}(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}));$$
2. for all $\mathcal{M}, \mathcal{N} \models T$ and $A \subseteq M$, if $f : A \to N$ is a homomorphism, $\bar{a} \in A$ and $\mathcal{M} \models \phi(\bar{a})$, then, $\mathcal{N} \models \phi(f(\bar{a}))$.

Definition 2. A **Boolean algebra** is a structure $(B; , \wedge, \vee, \neg, 0, 1)$, where $\wedge, \vee$ are binary operations, $\neg$ is unary, and 0 and 1 are constants naming distinct elements satsifying the following universal axioms (for all x, y, z):

- (de Morgan's laws) $\neg(\neg x) = x,\ \neg(x \wedge y) = \neg x \vee \neg y,\ \neg(x \vee y) = \neg x \wedge \neg y$;
- (associativity of $\wedge$): $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
- (associativity of $\vee$): $(x \vee y) \vee z = x \vee (y \vee z)$;
- (distributivity of $\wedge$ over $\vee$): $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
- (distributivity of $\vee$ over $\wedge$): $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
- (commutativity) $x \wedge y = y \wedge x,\ x \vee y = y \vee x$;
- $x \wedge \neg x = 0,\ x \vee \neg x = 1$;
- $x \wedge 0 = 0,\ x \vee 0 = x, x \wedge 1 = x.x \vee 1 = 1$;
- $0 \neq 1, \neg 0 = 1, \neg 1 = 0$.

Definition 3. In a Boolean algebra, the relation $x \wedge y = x$ defines a partial order which we denote by $x \leq y$. An atom in a Boolean algebra is a non-zero element a such that the only elements $\leq a$ are 0 and a. We say that B is **atomless** if it has no atoms.

EXERCISE 2. Note that any atomless Boolean algebra must be infinite. Prove that the theory of atomless Boolean algebras has quantifier elimination and is ω-categorical (and thus complete).

EXERCISE 3. Consider the theory $T = \mathrm{Th}(\mathbb{N}; s)$ where s is the successor operation $s(x) = x + 1$. Show that this theory is categorical in all uncountable cardinals.

EXERCISE 4. Let $\mathcal{L}_n$ be the language with n-many unary predicate symbols $P_1, \ldots, P_n$. Let T_n be the theory asserting each P_i is infinite, that they are all disjoint and there are infinitely many elements not in any P_i for $i \leq n$.

- Show T_n is ω-categorical and complete;
- How many models of cardinality $\aleph_1$ does T_2 have up to isomorphism?
- Show that $T = \bigcup_{n\in\mathbb{N}} T_n$ is a complete theory. How many countable models does T have up to isomorphism?

Definition 4. A **derivation** on a ring R is an additive homomorphism $D : R \to R$ such that

$$D(xy) = xD(y) + yD(x).$$

A **differential ring** is a ring equipped with a derivation. Given a derivation $D_0 : R \to R$, we call the **ring of differential polynomials** the differential ring

$$R\{X\} := R[X, X^{(1)}, X^{(2)}, X^{(3)}, \ldots],$$

with derivation D extending D_0 by setting $D\left(X^{(n)}\right) = X^{(n+1)}$.

Definition 5. An ideal $I \subseteq R\{X\}$ is a **differential ideal** if for all $f \in I, D(f) \in I$.

Definition 6. If $f(X) \in R\{X\} \setminus R$, the **order** of f is the largest n such that $X^{(n)}$ occurs in f. We can write

$$f(X) = \sum_{i=0}^{m} g_i(X, X^{(1)}, \ldots, X^{(n-1)})(X^{(n)})^i,$$

for $g_i \in R[X, X^{(1)}, \ldots, X^{(n-1)}]$ and $g_m \neq 0$. We call m the **degree** of f. We write $g \leq f$ if either the order of g is strictly less than the order of f or if the orders are the same but g has lower degree than f.

Fact 7. *Let K be a differential field of characteristic zero (in the sense that the underlying field has characteristic zero). Let I be a non-zero prime differential ideal. Then, there is $f \in I$ irreducible such that for all $g \in I$ with $g \neq 0$, $g \not\leq f$.*

Definition 8. A differential field K (of characteristic zero) is called **differentially closed** if for any non-constant differential polynomials f and g where the order of g is less than the order of f there is an x such that $f(x) = 0$ and $g(x) \neq 0$.

Note that the theory of differentially closed fields of characteristic zero is axiomatisable. We call it DCF_0.

Fact 9. *Every differential field k has an extension K which is differentially closed.*

EXERCISE 5. Prove that DCF_0 has quantifier elimination and is complete.

$\star\star$ **EXERCISE 6.** Prove the **differential Nullstellensatz** (everything is in characteristic 0): Suppose that k is a differential field and Σ is a finite system of differential equations and inequations over k which has a solution in some $l \supseteq k$. Then, Σ has a solution in any differentially closed $K \supseteq k$.