Extra exercises are marked with a $\star\star$. I DO NOT EXPECT YOU TO ANSWER THEM. I hope they can bring you joy.

Definition 1. Let $A \subseteq \mathcal{M} \models T$. We say that $\mathcal{M}$ is **prime over** A if for all $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ a partial elementary map, f extends to an elementary $f' : \mathcal{M} \to \mathcal{N}$.

EXERCISE 1. Show the following: Let T be a countable ω-stable theory, $\mathcal{M} \models T$ and $A \subseteq \mathcal{M}$. Then, there is $\mathcal{M}_0 \preceq \mathcal{M}$ which is a prime model over A and such that every $a \in M_0$ realises an isolated type over A.

Theorem 2 (Lachlan). *Let T be ω-stable, $\mathcal{M} \models T$, $|M| \geq \aleph_1$. Then, for each $\kappa > |M|$ there is $\mathcal{N} \succeq \mathcal{M}$ of cardinality κ such that for any countable set of $\mathcal{L}(M)$-formulas $\Gamma(x)$ in a finite variable x, if $\mathcal{N}$ realises $\Gamma(x)$, then so does $\mathcal{M}$.*

EXERCISE 2. We shall prove Theorem 2 following the steps below. Consider an ω-stable theory T and $\mathcal{M} \models T$, such that $|M| \geq \aleph_1$. Say that an $\mathcal{L}(M)$-formula is **large** if $\phi(M)$ is uncountable.

- Prove that there is a large $\mathcal{L}(M)$-formula $\phi_0(x)$ such that for any other $\mathcal{L}(M)$-formula ψ, either $\phi_0(x) \wedge \psi(x)$ or $\phi_0(x) \wedge \neg\psi(x)$ has a countable set of realisations.
- Consider
$$p(x) := \{\psi(x) | \psi(x) \in \mathcal{L}(M) \text{ and } \phi_0(x) \wedge \psi(x) \text{ is large }\}.$$
Show that p is a complete type over M which is not realised in M but such that all of its countable subsets are realised in M. Take $\mathcal{N}' \succeq \mathcal{M}$ with a point a realising p.
- By Exercise 1, take $\mathcal{N} \preceq \mathcal{N}'$ prime over Ma and such that every $b \in \mathcal{N}$ realises an isolated type over Ma. Show that for every $b \in N$, every countable subset $\Gamma(x)$ of $\mathrm{tp}(b/M)$ is realised in M.
- Deduce Theorem 2.

EXERCISE 3. Show that the theory of the random graph has a Vaughtian pair.

$\star\star$ **EXERCISE 4.** Show that there is no Vaughtian pair of real closed fields.

Definition 3. We say that T **eliminates the quantifier** $\exists^\infty x$ if for every $\mathcal{L}$-formula $\phi(x, \overline{y})$ there is $n_\phi \in \mathbb{N}$ such for all tuples $\overline{a} \in \mathbb{M}^{|\overline{y}|}$, if $|\phi(\mathbb{M}, \overline{a})| \geq n_\phi$, then $\phi(\mathbb{M}, \overline{a})$ is infinite.

EXERCISE 5. Show that if T has no Vaughtian pairs, then it eliminates the quantifier $\exists^\infty x$.

EXERCISE 6. Suppose that T eliminates the quantifier $\exists^\infty x$. Let $\mathcal{M} \models T$ and let $\phi(x) \in \mathcal{L}(M)$ be minimal in $\mathcal{M}$. Show that $\phi(x)$ is strongly minimal.

Definition 4. For infinite cardinals $\kappa > \lambda$, we say that T has has a (λ, κ)-model if $|M| = \kappa$ and for some $\phi(x) \in \mathcal{L}$, $|\phi(M)| = \lambda$.

EXERCISE 7. Prove the following:

1. If T has a (κ, λ)-model then it has a Vaughtian pair (and so an $(\aleph_1, \aleph_0)$-model [Hint: this should be trivial];
2. Prove that if T is ω-stable and has an $(\aleph_1, \aleph_0)$-model, then for each $\kappa > \aleph_1$, T has a $(\kappa, \aleph_0)$-model [Hint: you may need to use Theorem 2].

$\star\star$ **EXERCISE 8.** We show that in Exercise 7 (2), the assumption of ω-stability is necessary. Let $\mathcal{L} = \{P_0, \ldots, P_n, E_1, \ldots, E_n\}$ for unary predicates P_i and binary relations E_i. Consider the $\mathcal{L}$-theory T stating that:

- the P_i are infinite and partition the domain;
- for each $i \in \{1, \ldots, n\}$, $\forall xy(E_i(x, y) \to P_{i-1}(x) \wedge P_i(y))$;

- for each $i \in \{1, \ldots, n\}$, $\forall xy((P_i(x) \wedge P_i(y) \wedge \forall z(E_i(z,x) \leftrightarrow E_i(z,y)) \rightarrow x = y)$.

For example, for X_0 an infinite, take $X_{i+1} = \mathcal{P}(X_i)$ for $i \in \{1, \ldots, n\}$. Let $\mathcal{M}$ be the disjoint union of the X_i with P_i naming each of the X_i and E_i being the membership relation restricted to $X_i \times x_{i+1}$. Then, $\mathcal{M} \models T$. Show that if $\mathcal{M} \models T$ and $|P_0(M)| = \aleph_0$, then $|M| \leq \beth_n$. Hence, $\mathcal{M}$ has a $(\beth_n, \aleph_0)$-model but it does not have a $(\kappa, \aleph_0)$-model for arbitrarily large κ. [Hint: I would only do the case of $n = 1$. Recall that $\beth_0 = \aleph_0$ and $\beth_{\alpha+1} = 2^{\beth_\alpha}$.]