EXERCISE 1. 1. Prove that $(\mathbb{N}; 0, 1, +, \times, <)$ is the prime model of $\mathrm{Th}(\mathbb{N}; 0, 1, +, \times, <)$;

2. Prove that $\mathrm{Th}(\mathbb{Z}, 0, +)$ does not have a prime model. [Hint: you may assume that augmenting $(\mathbb{Z}, 0, +)$ by predicates P_n for each $n \geq 2$ naming the elements divisible by n yields a theory with quantifier elimination];

Definition 1. Let κ be an infinite cardinal. We say that T is **κ-stable** if for all A such that $|A| \leq \kappa$, $|S_x(A)| \leq \kappa$. We say that T is **superstable** if it is κ-stable for all $\kappa \geq 2^{|T|}$. We say that T is **stable** if it is κ-stable for some κ.

EXERCISE 2. Let $\mathcal{L} := \{U_i | i < \omega\}$ be such that each U_i is a unary predicate. For X and Y disjoint finite subsets of $\mathbb{N}$, let $\phi_{X,Y}$ be the sentence

$$\exists x \bigwedge_{i \in X} U_i(x) \wedge \bigwedge_{i \in Y} \neg U_i(x).$$

Let $T := \{\phi_{X,Y} | X, Y \text{ disjoint finite subsets of } \mathbb{N}\}$. You may assume this theory is complete and has quantifier elimination.

1. Show that no type over $\emptyset$ is isolated. Deduce that T has no prime models;
2. Show that T is κ-stable for all $\kappa \geq 2^{\aleph_0}$.

0.0.1 Equivalence relations

Example 2. Consider the following theories of equivalence relations for α an ordinal and κ a cardinal.

- **Refining equivalence relation with infinite splitting:** REI_α has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, E_j refines E_i and each E_i class is refined into infinitely many E_{i+1}-classes. For this and all other examples below we assume each equivalence class of each equivalence relation is infinite;
- **Refining equivalence relation with finite splitting:** REF_α has equivalence relations $(E_i | i < \alpha)$ such that for $i < j < \alpha$, E_j refines E_i and each E_i class is refined into two E_{i+1}-classes;
- **Crosscutting Equivalence relation with finite splitting:** CEF_κ has equivalence relations $(E_i | i < \kappa)$ such that each E_i has only two classes and for all $i < \kappa$, E_{i+1} splits each equivalence class of E_i into two classes;
- **Crosscutting Equivalence relation with infinite splitting:** CEI_κ has equivalence relations $(E_i | i < \kappa)$ such that each E_i has infinitely many classes and for all $i < \kappa$, E_{i+1} splits each equivalence class of E_i into infinitely many classes. These theories are complete and have quantifier elimination.

EXERCISE 3. Show the following:

1. REI_α is stable. REI_ω is κ-stable if and only if $\kappa^{\aleph_0} = \kappa$. [More generally, if α is infinite, REI_α is not superstable];
2. REF_α is stable. If $\alpha \leq \omega$ it is superstable. If $\alpha \geq \omega \cdot \omega$ then it is not superstable;
3. if $\kappa \leq \omega$, then CEF_κ is superstable.

$\star\star$ **EXERCISE 4.** Compute the number of models of each of the examples in Example 2 in each cardinality.

0.0.2 Differentially closed fields

Definition 3. A **derivation** on a ring R is an additive homomorphism $D : R \to R$ such that

$$D(xy) = xD(y) + yD(x).$$

A **differential ring** is a ring equipped with a derivation. Given a derivation $D_0 : R \to R$, we call the **ring of differential polynomials** the differential ring

$$R\{X\} := R[X, X^{(1)}, X^{(2)}, X^{(3)}, \ldots],$$

with derivation D extending D_0 by setting $D\left(X^{(n)}\right) = X^{(n+1)}$.

Definition 4. An ideal $I \subseteq R\{X\}$ is a **differential ideal** if for all $f \in I, D(f) \in I$.

Definition 5. If $f(X) \in R\{X\} \setminus R$, the **order** of f is the largest n such that $X^{(n)}$ occurs in f. We can write

$$f(X) = \sum_{i=0}^{m} g_i(X, X^{(1)}, \ldots, X^{(n-1)})(X^{(n)})^i,$$

for $g_i \in R[X, X^{(1)}, \ldots, X^{(n-1)}]$ and $g_m \neq 0$. We call m the **degree** of f. We write $g \leq f$ if either the order of g is strictly less than the order of f or if the orders are the same but g has lower degree than f.

Definition 6. A differential field K (of characteristic zero) which is algebraically closed is called **differentially closed** if for any non-constant differential polynomials f and g where the order of g is less than the order of f there is an x such that $f(x) = 0$ and $g(x) \neq 0$.

Note that the theory of differentially closed fields of characteristic zero is axiomatisable. We call it DCF_0.

Fact 7. *Every differential field k has an extension K which is differentially closed.*

Fact 8. DCF_0 *is complete and has quantifier elimination.*

$\star\star$ **EXERCISE 5.** Let $K \models \mathrm{DCF}_0$. Let $k \subseteq K$.

1. for $p \in S_1^K(k)$, the set of 1-types over k realised in K, let $I_p := \{f \in k\{X\} | f(x) = 0 \in p\}$. Show that I_p is a differential prime ideal [i.e. $I_p \subsetneq k\{X\}$ and if $f \cdot g \in I_p$, then either $f \in I_p$ or $g \in I_p$].
2. Show that if $I \subset k\{X\}$ is a differential prime ideal, then $I = I_p$ for some $p \in S_n^K$. So $p \mapsto I_p$ is a bijection between complete n-types and differential prime ideals in $k\{X\}$;
3. The Ritt-Raudenbusch Basis Theorem says that every differential prime ideal in $k\{X\}$ is finitely generated. Use this to show that DCF_0 is ω-stable;
4. Suppose that k is a differential field of characteristic zero and $k \subseteq K \models \mathrm{DCF}_0$. We say that K is the **differential closure** of k if for all $L \supseteq k$ such that $K \models \mathrm{DCF}_0$ there is a differential field embedding of K into L fixing k. Show that every field has a differential closure.