Definition 1. We say that $\phi(x, y)$ has the **independence property**, IP, if there are $(b_i | i < \omega)$ and $(a_s | s \subseteq \omega)$ such that

$$\vDash \phi(a_s, b_i) \Leftrightarrow i \in s.$$

We say that $\phi(x, y)$ is NIP if it does not have the independence property. A theory is NIP if all of its formulas are NIP.

Example 2. The random graph has the independence property.

EXERCISE 1. Prove that a formula $\phi(x, y)$ has IP if and only if there is an indiscernible sequence $(b_i | i \in I)$ and a parameter a such that

$$\vDash \phi(a, b_i) \text{ if and only if } i \text{ is even.}$$

Definition 3. We say that $\phi(x, y)$ has the **strict order property**, SOP if there is an infinite sequence $(b_i)_{i<\omega}$ such that $\phi(\mathbb{M}, b_i) \subsetneq \phi(\mathbb{M}, b_j)$ for all $i < j < \omega$. We say that $\phi(x, y)$ is NSOP if it does not have the strict order property, and we say that a theory T is NSOP if all of its formulas are NSOP.

EXERCISE 2. Prove that T is stable if and only if it is NIP and NSOP. [Hint: For the $(\Leftarrow)$ direction, consider an unstable theory with NIP and prove it must have the SOP.]

Definition 4. Let R be an associative ring with identity. Let $\mathcal{L}_R := \{+, 0, (\mu_r)_{r \in R))}\}$, where the μ_r are unary function symbols. The theory of (left) R-modules T_R asserts of a model M that $(M, +, 0)$ is an Abelian group, and each f_r is an endomorphism of this group with f_1 being the identity map. Note that this theory is not complete.

Definition 5. An $\mathcal{L}$-formula is called **primitive positive** if it is an existential quantification of a conjunction of atomic formulas. That is, if it is of the form

$$\exists \overline{y} \bigwedge_{i \leq n} \psi_i(\overline{x}, \overline{y}),$$

where the formulas $\psi_i(\overline{x}, \overline{y})$ are atomic.

Fact 6. *The theory T_R has elimination of quantifiers up to primitive positive formulas. That is, for every $\mathcal{L}_R$-sentence, $\phi(\overline{x})$ there is a Boolean combination of primitive positive formulas $\psi(\overline{x})$ such that $T_R \vDash \forall \overline{x} \phi(\overline{x}) \leftrightarrow \psi(\overline{x})$.*

EXERCISE 3. Consider a primitive positive $\mathcal{L}_R$-formula $\phi(\overline{x}, \overline{z})$ and let M be an R-module. Show that $\phi(\overline{x}, \overline{0})$, where $\overline{0}$ is a string of 0's of length $|\overline{z}|$ defines either $\emptyset$ or a subgroup of $M^{|\overline{x}|}$. Show that for any tuple $\overline{a}$ such that $\phi(\overline{x}, \overline{a})$ is consistent, $\phi(M, \overline{a})$ is a coset of $\phi(\overline{x}, \overline{0})$. Deduce that that for any tuples $\overline{a}, \overline{b}$, $\phi(\overline{x}, \overline{a})$ and $\phi(\overline{x}, \overline{b})$ are either mutually inconsistent or equivalent. Using elimination of quantifiers up to pp-formulas, prove that for any R-module M, its theory is stable.

Definition 7. A group G satisfies the **ω-stable descending chain condition** if there is no infinite proper descending chain of definable subgroups of G,

$$\cdots < H_{i+1} < H_i < \cdots < H_1 < G.$$

EXERCISE 4. Show that if G is an ω-stable group (i.e. a group whose theory is ω-stable), then G satisfies the ω-stable descending chain condition. Show that for an R-module M, the following are equivalent:

- M is ω-stable;
- M has the ω-stable descending chain condition;
- For any set of pp-formulas in a single variable $\{\phi_n(x) | n < \omega\}$ there is some $n \in \omega$ such that for all $m \in \omega$, $M \vDash \forall x (\bigwedge_{i<n} \phi_i(x) \rightarrow \phi_m(x))$.