When invariance implies exchangeability

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① Exchangeable graphs

An **exchangeable graph** is a Borel probability measure on the space of graphs on $\mathbb{N}$ invariant under the action of S_∞ (i.e. all permutations).

Example: construction of the random graph by tossing coins for each pair of points in $\mathbb{N}$.

They are studied in:

- **Probability:** Aldous & Hoover representation theorem for exchangeable graphs [3];
- **Statistical networks:** natural setting to think probabilistically about a graph on a large population with no extra information;
- **Combinatorics:** correspond to graphons;
- **Logic:** [1] answer: for what $\mathcal{N}$ is there an exchangeable structure concentrating on its isomorphism type?

② Invariant Random Expansions

$\mathcal{C}'$ a hereditary class of finite relational structures.

An **invariant random expansion (IRE)** of $\mathcal{M}$ by $\mathcal{C}'$ is an $\mathrm{Aut}(M)$-invariant regular Borel probability measure on the space of structures $\mathcal{N}'$ with domain M and age (i.e. class of finite substructures) contained in $\mathcal{C}'$.

IREs of $(\mathbb{N},=)$ are **exchangeable structures**.

We consider $\mathcal{M}$ countable and **homogeneous:** any isomorphism between finite substructures extends to an automorphism.

③ Previous work

Previous work characterised IREs for:

- $\mathcal{M}$ very well-behaved: $(\mathbb{Q},<)$ [9], or with disjoint n-amalgamation (i.e. no interesting omitted substructures) [6];
- $\mathcal{C}'$ with very slow growth rate: unary structures, or linear orders [8, 4].

We want to understand IREs of $\mathcal{M}$ with interesting omitted substructures.

When are all IREs of $\mathcal{M}$ by $\mathcal{C}'$ exchangeable?

- IREs of $\mathcal{M}$ by $\mathcal{C}'$ always include the exchangeable ones, which are well-understood;

④ The Main Theorem

Let $k \geq 1$ and $\mathcal{M}$ be homogeneous with **k-overlap closed** age.
Let $\mathcal{C}'$ have labelled growth rate $O(e^{n^{k+\delta}})$ for every $\delta > 0$.
Then every IRE of $\mathcal{M}$ by $\mathcal{C}'$ is exchangeable.

Example: IREs of the generic tetrahedron-free 3-hypergraph by graphs are exchangeable.

Proof uses probabilistic method techniques inspired by [4].
Works for **Consistent Random Expansions**: random expansions of an hereditary class $\mathcal{C}$ by another $\mathcal{C}'$.

⑤ k-overlap closed examples

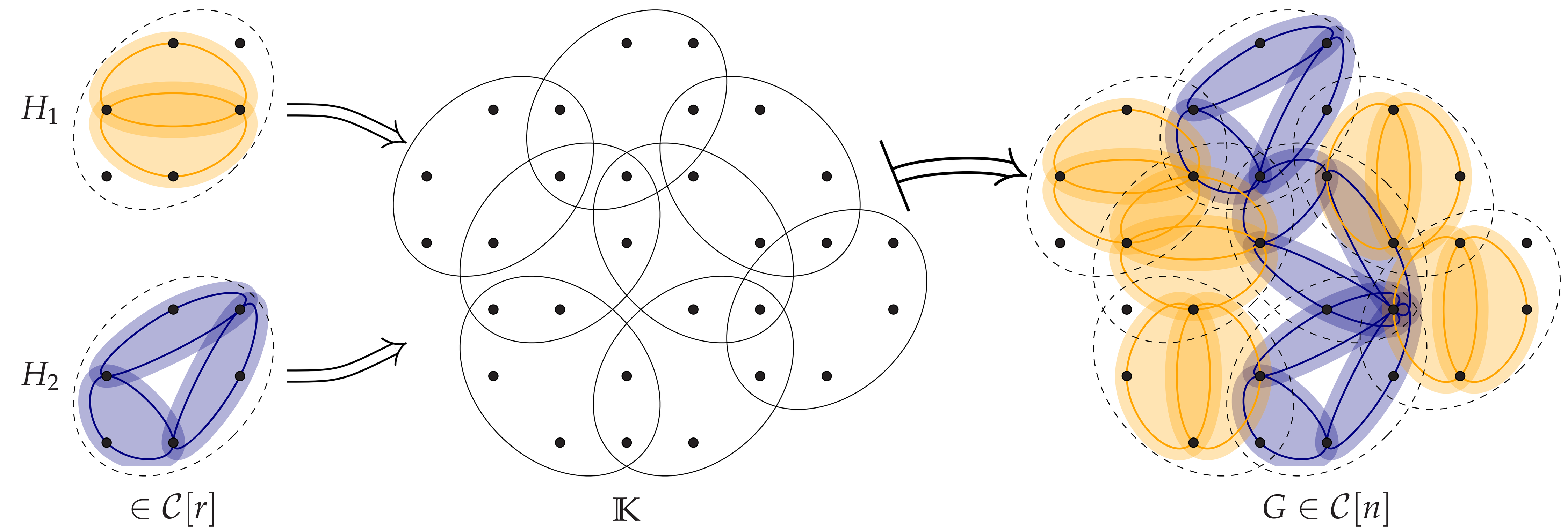
- **1-overlap closed:** free amalgamation classes in arity > 1;
- **2-overlap closed:** 3-hypergraphs, tetrahedron-free 3-hypergraphs;
- **k-overlap closed:** $\mathcal{C} = \mathrm{Forb}(\mathcal{F})$ of arity $> k$ and all $A \in \mathcal{F}$ are $(k+1)$-irreducible, or of bounded size and k-irreducible (for $k \geq 2$)

k-irreducible: every k-many vertices are related.

⑥ Growth rate examples

- $O(e^{n^{1+\delta}})$: unary structures, the age of any NIP homogeneous structure;
- $O(e^{n^{2+\delta}})$: graphs;
- $O(e^{n^{k+\delta}})$: structures with finitely many k-ary relations.

⑦ k-overlap closedness



$\mathcal{L}$ of arity $> k$. $\mathcal{C}$ is **k-overlap closed** if for every $r > k$ and arbitrarily large n, there exists an r-uniform hypergraph $\mathbb{K}$ on n vertices s.t.

1. **(Density)** $\mathbb{K}$ has at least $C(r)n^{k+\alpha(r)}$ hyperedges for some $\alpha(r) > 0$;
2. **(Pasting)** For every $H_1, H_2 \in \mathcal{C}[r]$, pasting them into the $\mathbb{K}$-hyperedges yields $G \in \mathcal{C}[n]$ (possibly after adding extra relations).

⑧ Moral of the story

If $\mathcal{M}$ is k-transitive and "looks random enough", IREs by "essentially k-ary" classes are exchangeable.

If $\mathcal{M}$ has "hidden k-ary structure", it can have non-exchangeable IREs by $\mathcal{C}'$ of growth rate $O(e^{n^{k+\delta}})$:

- NIP homogeneous structures;
- universal homogeneous kay-graphs.

⑨ Invariant Keisler measures

An **Invariant Keisler measure (IKM)** is an $\mathrm{Aut}(M)$-invariant regular Borel probability measure on $S_x(M)$.

IKMs are a special case of IREs!

- By the Main Theorem, IKMs of the generic tetrahedron-free 3-hypergraph, and of many other homogeneous structures are S_∞-invariant!
- Previous work only understood IKMs for homogeneous $\mathcal{M}$ binary or NIP [2, 7];
- We give $2^{\aleph_0}$ very tame simple homogeneous structures with universally measure zero nonforking formulas (cf. [5]).

References

[1] N. Ackerman, C. Freer, and R. Patel. "Invariant measures concentrated on countable structures". In: *Forum Math. Sigma.* 2016.

[2] M. H. Albert. "Measures on the random graph". In: *J. Lon. Math. Soc.* (1994).

[3] D. J. Aldous. "Representations for partially exchangeable arrays of random variables". In: *J. Mult. Anal.* 11.4 (1981).

[4] O. Angel, A. S. Kechris, and R. Lyons. "Random orderings and unique ergodicity of automorphism groups". In: *JEMS* (2014).

[5] A. Chernikov, E. Hrushovski, A. Kruckman, K. Krupiński, S. Moconja, A. Pillay, and N. Ramsey. "Invariant measures in simple and in small theories". In: *J. Math. Log.* (2023).

[6] H. Crane and H. Towsner. "Relatively exchangeable structures". In: *J. Symb. Log.* (2018).

[7] D. E. Ensley. "Automorphism-invariant measures on $\aleph_0$-categorical structures without the independence property". In: *J. Symb. Log.* (1996).

[8] C. Jahel and T. Tsankov. "Invariant measures on products and on the space of linear orders". In: *J. Ecole Pol. Math.* (2022).

[9] O. Kallenberg. *Probabilistic symmetries and invariance principles.* Springer, 2005.